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On Gradient Ricci Soliton Riemannian Submersions

Adrian Vinícius Castro Ribeiro

Manaus, Amazonas, Brazil
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by

Adrian Vinícius Castro Ribeiro

Advisor

Prof. Ph.D. José Nazareno Vieira Gomes

Co-Advisor

Prof. Ph.D. Marcus Antônio Mendonça Marrocos

Thesis submitted to the Graduate Program in Mathematics in Association with the Universidade Federal do Pará and Universidade Federal do Amazonas in partial fulfillment of the necessary requirements for the degree of Ph.D. in Mathematics.
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Resumo

Nesta tese nós mostramos como construir sólitons de Ricci gradientes que são realizados como submersões Riemannianas com espaço total tendo fibras totalmente umbílicas e distribuição horizontal integrável. Esta construção é baseada em uma generalização de produtos deformados para fibrados, bem como, em uma construção de sólitons de Ricci gradiente produtos deformados a partir do qual nós sabemos que os espaços base de tais produtos deformados são necessariamente variedades tipo Ricci-Hessiano. Ao estudar esta última classe de variedades Riemannianas nós também obtemos resultados de trivialidade e inexistência de sólitons de Ricci gradiente produtos deformados. Estes resultados decorrem de um teorema tipo Liouville e da validade de um princípio do máximo fraco no infinito para um operador de difusão específico sobre uma variedade tipo Ricci-Hessiano.

Palavras-chave: Sóliton de Ricci; Submersão Riemanniana; Métrica tipo Einstein; Produto deformado.

Abstract

In this thesis we show how to construct gradient Ricci solitons that are realized as Riemannian submersions with total space having totally umbilical fibers and integrable horizontal distribution. This construction is based on a generalization of warped products to bundles as well as a construction of gradient Ricci soliton warped products, from which we know that the base spaces of such warped products are necessarily Ricci-Hessian type manifolds. By studying this latter class of Riemannian manifolds we also obtain triviality and nonexistence results for gradient Ricci soliton warped products. These results stem from a Liouville type theorem and the validity of a weak maximum principle at infinity for a specific diffusion operator on a Ricci-Hessian type manifold.

Keywords: Ricci soliton; Riemannian submersion; Einstein type metric; Warped product.

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Introduction

This thesis is divided in two chapters. Chapter 1 is about a construction of gradient Ricci solitons on the total space M of a Riemannian submersion. A complete Riemannian metric g on a smooth manifold M is a *gradient Ricci soliton* if there exists a smooth function Ψ on M such that the Ricci tensor of g is given by

$$\operatorname{Ric} + \nabla^2 \Psi = \lambda g, \tag{1}$$

for some constant $\lambda \in \mathbb{R}$. Note that the parameters in Eq. (1) are g and Ψ, while the constant λ is obtained by taking trace of this equation. A gradient Ricci soliton is called *expanding*, *steady* or *shrinking* if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. When Ψ is a constant function (M, g) is an Einstein manifold and it is called a *trivial Ricci soliton.*

Our purpose is to establish the necessary and sufficient conditions for a complete Riemannian metric g on M be a gradient Ricci soliton with potential function $\Psi = \tilde{\varphi}$ so that $\pi : (M, g) \to (B, g_B)$ be a Riemannian submersion on a Riemannian manifold (B, g_B) for some smooth function φ on B. This choice of potentials functions as horizontal lifts is motivated by the warped product case, as will be seen in Chapter 2. For simplicity, we write $(\pi, g, \tilde{\varphi})$ to denote a gradient Ricci soliton on the total space of π with potential function $\tilde{\varphi} = \pi \circ \varphi$.

We show how to construct a gradient Ricci soliton $(\pi, g, \tilde{\varphi})$ with totally umbilical fibers and integrable horizontal distribution. This construction stems from two known works. The first one is based on the construction of gradient Ricci soliton warped products by Feitosa et al. [4]. Authors proved that if $f > 0$ and φ are smooth functions on a Riemannian manifold (B, g_B) such that

$$\operatorname{Ric} + \nabla^2 \varphi = \lambda g_B + \frac{m}{f} \nabla^2 f \quad \text{and} \quad 2\lambda\varphi - |\nabla\varphi|^2 + \Delta\varphi + \frac{m}{f} \nabla\varphi(f) = c, \tag{2}$$

for some constants $\lambda, m, c \in \mathbb{R}$, with $m \neq 0$, then they must satisfy

$$\lambda f^2 + f\Delta f + (m-1)|\nabla f|^2 - f\nabla\varphi(f) = \mu, \tag{3}$$

for some constant $\mu \in \mathbb{R}$, see [4, Proposition 3]. Equations (2) and (3) will be studied in Chapter 2 in more detail. The second work is based on the generalization of warped products to bundles due to Bishop and O'Neill [14]. They considered two Riemannian manifolds (B, g_B) and (F, g_F) and they showed how to construct a fiber bundle structure $F \to M \xrightarrow{\pi} B$ whose structural group is $\pi_1(B)$ and total space $M = \tilde{B} \times_{\pi_1(B)} F$ having integrable horizontal distribution and totally geodesic fibers. Moreover, by use of a smooth function $f > 0$ on B they further warped the standard quotient metric g on M.

Making use of the previous construction and with the aforementioned notations we show how to construct a gradient Ricci soliton Riemannian submersion as follows.

Theorem 1 *Let (B, g_B) be a complete Riemannian manifold with two smooth functions f and φ satisfying Eq.* (2)*, for any $\lambda \in \mathbb{R}$. Take the constant μ given by Eq.* (3) *and a complete Riemannian manifold (F, g_F) of dimension m and Ricci tensor $\mathrm{Ric}_F = \mu g_F$. Then, we can construct a gradient Ricci soliton $(\pi, \bar{g}, \tilde{\varphi})$ with total space $\tilde{B} \times_{\pi_1(B)} F$ having totally umbilical fibers and integrable horizontal distribution, where $\bar{g}$ is a warped metric which is obtained from the standard quotient metric g.*

A trivial example agreeing with Theorem 1 is well known in the literature and has been explored in a context of rigidity, see Fernández-López and García-Río [10] as well as Petersen and Wylie [12]. A gradient Ricci soliton satisfying (1) is said to be *rigid* if it is isometric to a quotient $\mathbb{R}^n \times_\Gamma F^m$, where F is an Einstein manifold with Einstein constant λ, the potential function is $\psi(x) = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^n$ and Γ is a group acting freely on F and by orthogonal transformations on $\mathbb{R}^n$. A nontrivial example in the setting of Theorem 1 has been explicitly constructed for the case of Kähler metrics by Huai-Dong Cao [7].

In Chapter 2 we study gradient Ricci solitons that are realized as warped products $M = B^n \times_f F^m$. We assume without loss of generality that the potential function of such a soliton is the lift $\tilde{\varphi}$ of a smooth function φ on B to M, see Lemma 2.1. Throughout the chapter ψ stands for the smooth function $\psi = \varphi - m \ln f$ on B.

We describe now the main theorems of the second chapter. We begin with the following triviality result for the steady case.

Theorem 2 *Let $B \times_f F$ be a gradient steady Ricci soliton with fiber having nonnegative scalar curvature. Then, it must be a standard Riemannian product provided the warping function satisfies $f \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

We also prove a nonexistence theorem for the expanding case.

Theorem 3 *It is not possible to construct a gradient expanding Ricci soliton warped product $B^n \times_f F^m$ with fiber having nonnegative scalar curvature and warping function satisfying either of the following conditions: $f \in L^\infty(B)$ or $f \in L^p(B^n, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

The next result establishes conditions on the potential function for a gradient expanding Ricci soliton to be trivial.

Theorem 4 *Let $B^n \times_f F^m$ be a gradient expanding Ricci soltion with potential function $\tilde{\varphi}$. Then, $B^n \times_f F^m$ is a trivial Ricci soliton provided that φ satisfies either of the following conditions: $|\nabla\varphi| \in L^\infty(B)$ or $|\nabla\varphi| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

We also prove a nonexistence result in the shrinking case.

Theorem 5 *It is not possible to construct a gradient shrinking Ricci soliton $B^n \times_f F^m$ with fiber having nonpositive scalar curvature and warping function satisfying either of the following conditions: $|\nabla \ln f| \in L^\infty(B)$ or $|\nabla \ln f| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

We point out that a class of gradient expanding Ricci soliton warped products with fiber having nonpositive scalar curvature has been constructed by Feitosa et al. [4, Corollary 2]. It is also known that Robert Bryant constructed a gradient steady Ricci soliton warped product with fiber having positive scalar curvature, see Chow et al. [3]. Thus, some assumption on the warping function is necessary to obtain triviality or nonexistence results in the class of gradient steady or expanding Ricci soliton warped products. Theorems 2 and 3 are analogous results to the Einstein warped product case proven by Rimoldi, see [11, Theorems 1, 9 and 11].

We begin in Section 2.1 with some comments and results on Ricci-Hessian type manifolds, which show us that this class of Riemannian manifolds is interesting in its own right, see Section 2.1. By studying such manifolds we use methods from weighted manifold theory to prove the validity of a weak maximum principle at infinity for a specific diffusion operator, see Proposition 2.4. In particular, such a principle is valid in the setting of gradient Ricci solitons as well as of m-quasi-Einstein manifolds as proven before by Pigola et al. [15] and by Rimoldi [11], respectively. To prove the main theorems we restrict ourselves to Ricci-Hessian type manifolds as being the base spaces of gradient Ricci soliton warped products, see Section 2.2. We finalize this thesis in Section 2.3 by computing scalar curvature estimates for Ricci-Hessian type manifolds. Furthermore, results of triviality and rigidity at the extreme values of the scalar curvature have been addressed.

Chapter 1

Gradient Ricci Soliton Riemannian Submersions

In this chapter we construct gradient Ricci solitons that are realized as a Riemannian submersion with total space having totally umbilical fibers and integrable horizontal distribution. We start by establishing the following preliminaries.

1.1 Preliminaries

Let (M, g) and (B, g_B) be Riemannian manifolds. Let $\pi : M \to B$ denote a smooth submersion, i.e., each derivative map π_* of π is surjective. Hence, for all $b \in B$, $F_b = \pi^{-1}(b)$ is a closed embedded submanifold of M which is called a *fiber*. For each $p \in M$, with $\pi(p) = b$, we denote by $\mathcal{V}_p$ the tangent space to F_b, by $\mathcal{H}_p$ the orthogonal complement of T_pF_b in T_pM and by g_{F_b} the restriction of g to F_b. We call $\mathcal{V}$ and $\mathcal{H}$ the vertical and horizontal distributions, respectively. The same letters will serve to denote the corresponding vertical and horizontal projections on the tangent bundle of M. Since $\mathcal{V}$ coincides with the tangential distribution defined by the fibers, the vertical distribution is integrable. But it is not necessarily true that the horizontal distribution $\mathcal{H}$ is integrable.

A vector field $E \in \mathfrak{X}(M)$ is *vertical* if $E_p \in \mathcal{V}_p$, and *horizontal* if $E_p \in \mathcal{H}_p$, for all $p \in M$. In this thesis U, V, W, W' stand for vertical fields, X, Y, Z, Z' for horizontal fields, and D stands for the Riemannian connection of g. It will be sufficient for our purposes to restrict our attention to *Riemannian submersions*, that is, to smooth submersions such that π_* preserves length of horizontal vector fields.

To understand the geometry of a Riemannian submersion in more detail we need to work with the following two tensors. The first one is the $(1,2)$-tensor T on $\mathfrak{X}(M)$ given by

$$T_{E_1}E_2 = \mathcal{H}(D_{\mathcal{V}E_1}\mathcal{V}E_2) + \mathcal{V}(D_{\mathcal{V}E_1}\mathcal{H}E_2).$$

This tensor satisfies the following properties:

1. $T_X U = T_X Y = 0$;
2. $T_U V = \mathcal{H}(D_U V)$ and $T_U X = \mathcal{V}(D_U X)$;
3. $T_U V = T_V U$;
4. T_U is alternating, that is, $g(T_U V, X) = -g(T_U X, V)$.

Notice that $T_U V$ is the second fundamental form of F_b, and therefore the tensor T vanishes identically if and only if each fiber F_b is totally geodesic.

The second tensor is the $(1,2)$-tensor A on $\mathfrak{X}(M)$ given by

$$A_{E_1}E_2 = \mathcal{H}(D_{\mathcal{H}E_1}\mathcal{V}E_2) + \mathcal{V}(D_{\mathcal{H}E_1}\mathcal{H}E_2).$$

The properties of A that we need are:

1. $A_U X = A_U V = 0$;
2. $A_X U = \mathcal{H}(D_X U)$ and $A_X Y = \mathcal{V}(D_X Y)$;
3. $A_X Y = -A_Y X$;
4. A_X is alternating, that is, $g(A_X Y, U) = -g(A_X U, Y)$;
5. $A_X Y = \frac{1}{2}\mathcal{V}\left([X,Y]\right)$.

The tensor A is related to the obstruction to integrability of $\mathcal{H}$. Indeed, it is identically zero if and only if $\mathcal{H}$ is integrable. We easily see that if $A \equiv 0$ then, at least locally, the total space M is isometric to $B \times F$ with a Riemannian metric $g_B + \bar{g}_b$, where $\bar{g}_b$ is a smooth family of Riemannian metrics on F indexed by B.

A vector field $E \in \mathfrak{X}(M)$ is *basic* if it is horizontal and if there exists a vector field $\bar{E}$ in B which is π-related to E, i.e., $\pi_*(E) = \bar{E}$. If π is surjective, then for every vector field $\bar{E} \in \mathfrak{X}(B)$, there exists one and only one basic vector field $E \in \mathfrak{X}(M)$ which is π-related to $\bar{E}$. In particular, if X and Y are basic vector fields, then $\mathcal{H}([X,Y])$ is the basic vector field π-related to $[\bar{X},\bar{Y}]$, and $\mathcal{H}(D_X Y)$ is the basic vector field π-related to ${}^B D_{\bar{X}}\bar{Y}$, where ${}^B D$ is the Riemannian connection of g_B. Observe that if X and Y are basic vector fields, then $g(X,Y) = g_B(\bar{X},\bar{Y})$ is constant on the fibers. A vector field U is vertical if and only if it is π-related to 0 in $\mathfrak{X}(B)$. Also, if X is a basic vector field and U is a vertical vector field, then $[X,U]$ is a vertical vector field.

We denote by ${}^{F_b}D$ the family of Riemannian connections of the metrics g_{F_b}. It follows from the uniqueness of ${}^{F_b}D$ that ${}^{F_b}D_U V = \mathcal{V}(D_U V)$.

We now summarize the relationships between T, A and D as follows:

1. $D_U V = {}^{F_b}D_U V + T_U V$;
2. $D_U X = T_U X + \mathcal{H}(D_U X)$;
3. $D_X U = \mathcal{V}(D_X U) + A_X U$;
4. $D_X Y = A_X Y + \mathcal{H}(D_X Y)$.

We denote by R the curvature tensor of g, by R_{F_b} the collection of all curvature tensors of the Riemannian metrics g_{F_b} in the fibers and by ${}^B R(X,Y)Z$ the horizontal vector field such that ${}^B R(X,Y)Z = R_B(\pi_* X, \pi_* Y)\pi_* Z$, where R_B is the curvature tensor of g_B. We recall at this juncture the following equations which will be exploited henceforth

1. $g(R(U,V)W,W') = g(R_{F_b}(U,W)W,W') - g(T_U W, T_V W') + g(T_V W, T_U W')$;
2. $g(R(U,V)W,X) = g((D_V T)_U W, X) - g((D_U T)_V W, X)$;
3. $$g(R(X,U)Y,V) = g((D_X T)_U V, X) - g(T_U X, T_V Y) + g((D_U A)_X Y, V) + g(A_X U, A_Y V);$$
4. $$g(R(U,V)X,Y) = g((D_U A)_X Y, V) - g((D_V A)_U X, Y) + g(A_X U, A_Y V) - g(A_X V, A_Y U) - g(T_U X, T_V Y) + g(T_V X, T_U Y);$$
5. $$g(R(X,Y)Z,U) = g((D_Z A)_X Y, U) + g(A_X Y, T_Z U) - g(A_Y Z, T_U X) - g(A_Z X, T_U Y);$$

6. $g(R(X,Y)Z,Z') = g({}^BR(X,Y)Z,Z') - 2g(A_XY, A_ZZ') + g(A_YZ, A_XZ') - g(A_XZ, A_YZ').$

To compute the Ricci curvature of (M, g, D), we take $p \in M$, an orthonormal basis $\{X_i\}_i$ for $\mathcal{H}_p$ and an orthonormal basis $\{U_j\}_j$ for $\mathcal{V}_p$, in such a way that we have the following special notations

$$
\begin{aligned}
(A_X, A_Y) &= \sum_i g(A_XX_i, A_YX_i) = \sum_j g(A_XU_j, A_YU_j);\\
(A_X, T_U) &= \sum_i g(A_XX_i, T_UX_i) = \sum_j g(A_XU_j, T_UU_j);\\
(AU, AV) &= \sum_i g(A_{X_i}U, A_{X_i}V);\\
(TX, TY) &= \sum_j g(T_{U_j}X, T_{U_j}Y).
\end{aligned}
$$

Moreover, for any tensor S on $\mathfrak{X}(M)$, one has

$$\check{\delta}S = -\sum_i (D_{X_i}S)_{X_i}, \quad \hat{\delta}S = -\sum_j (D_{U_j}S)_{U_j} \quad \text{and} \quad \delta S = \check{\delta}S + \hat{\delta}S.$$

Associated with T, one has a symmetric $(0,2)$-tensor $\tilde{\delta}T$ given by

$$(\tilde{\delta}T)(U,V) = \sum_i g((D_{X_i}T)_UV, X_i).$$

The mean curvature vector along each fiber is the horizontal vector field $N = \sum_j T_{U_j}U_j$. Notice that N vanishes identically if and only if each fiber is a minimal submanifold of M.

The *Yang-Mills condition* for $\mathcal{H}$ is defined by

$$g((\check{\delta}A)X, U) = (A_X, T_U), \quad \text{for} \quad X \in \mathcal{H}, \text{ and } U \in \mathcal{V}. \tag{1.1}$$

Introduced by Yang and Mills in physics, this condition was thoroughly studied in both mathematical physics and in pure mathematics. It is also important in the study of Einstein Riemannian submersions as we will see later. We also note that, since the tensor $\mathcal{V}(D_XA)_YZ - T_{A_YZ}X$ depends only on $\mathcal{H}$ and ${}^BD$, Eq. (1.1) depends only on $\mathcal{H}$ and g_B and does not depend on the family of the metrics g_{F_b} on the fibers.

Writing Ric, Ric_B and Ric_{F_b} for the Ricci curvatures of the metrics g, g_B and g_{F_b}, respectively, we have the expressions of the Ricci curvatures of a submersion which can be found in Besse [1]:

$$\begin{aligned}
\mathrm{Ric}(U,V) &= \mathrm{Ric}_{F_b}(U,V) - g(N,T_UV) + (AU,AV) + (\tilde{\delta}T)(U,V);\\
\mathrm{Ric}(X,U) &= g((\hat{\delta}T)U,X) + g(D_UN,X) + g((\check{\delta}A)X,U) - 2(A_X,T_U);\\
\mathrm{Ric}(X,Y) &= {}^B\mathrm{Ric}(X,Y) - 2(A_X,A_Y) - (T_X,T_Y) + \frac{1}{2}\left(g(D_XN,Y) + g(D_YN,X)\right).
\end{aligned}$$

Given a Riemannian submersion $\pi : (M,g) \to (B,g_B)$ and a smooth function f on B, we denote by $\nabla\tilde{f}$ and $\nabla^2\tilde{f}$ the gradient and the Hessian of $\tilde{f} = \pi^* f$ in (M,g) as well as by ${}^B\nabla f$ and ${}^B\nabla^2 f$ the gradient and the Hessian of f in (B,g_B), respectively. Direct computation shows that the vector field $\nabla\tilde{f}$ is horizontal and π-related to ${}^B\nabla f$. Furthermore, the following properties hold: $(\nabla^2\tilde{f})(U,V) = -(T_UV)\tilde{f}$ and $\nabla^2\tilde{f}(X,Y) = {}^B\nabla^2 f(\bar{X},\bar{Y})$. As a consequence we have $\Delta\tilde{f} = \pi^*\Delta_B f - N\tilde{f}$. Given a function f as described as above, we denote by H^f the horizontal lift of its Hessian tensor.

1.2 Submersions with totally geodesic fibers

Let M and B be connected Riemannian manifolds, and let $\pi : M \to B$ be a surjective Riemannian submersion not necessarily with connected fibers. Given a path γ in B, a *horizontal lift* of γ is any horizontal path c in M such that $\pi \circ c = \gamma$. Such lifts always exist, at least locally. To understand the global case we recall the following definition. Let $\mathcal{D}$ be a distribution on M which is a complement to the vertical distribution $\mathcal{V} = \ker(\pi_*)$. We say that $\mathcal{D}$ is *(Ehresmann-)complete* if for any path γ in B starting from $b \in B$, and for any $p \in F_b$ there exists a horizontal lift c of γ starting from p.

It will be sufficient to work with surjective submersions having complete total space. It is worth mentioning that if M is complete, then both $\mathcal{H}$ and B are complete, see [2, Proposition 2.1]. Moreover, if $\mathcal{H}$ is complete, then given any path $\gamma : [0,l] \to B$ and any $p \in F_{\gamma(0)}$, there exists a unique horizontal lift c of γ starting from p, which allows us to consider the diffeomorphism $\tau_\gamma : F_{\gamma(0)} \to F_{\gamma(l)}$ defined by setting $\tau_\gamma(p) = c(l)$, see [1, Proposition 9.30] and [2, Eq. (2.2)]. The *holonomy group* G_b of the connection $\mathcal{H}$ at $b \in B$ is the group of all diffeomorphisms τ_σ of F_b corresponding to the closed paths σ in B starting from b.

By the previous section we know that the fibers of a Riemannian submersion are totally geodesic if and only if $T = 0$. To give some motivation of where this study comes from, let us look back at the following result, in which the completeness of $\mathcal{H}$ implies that fibers become geometrically indistinguishable from one another.

Proposition 1.1 (R. Hermann [13]) *Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. If $\mathcal{H}$ is complete, then for any path $\gamma : [0, l] \to B$ the diffeomorphism τ_γ is an isometry and for all $b \in B$ the holonomy group G_b is a subgroup of the isometry group of (F_b, g_{F_b}).*

In this latter setting, Vilms showed how to construct a Riemannian submersion with totally geodesic fibers.

Proposition 1.2 (J. Vilms [8]) *Let G be a Lie group, $p : P \to B$ a principal G-bundle, F any manifold on which G acts. Let $\pi : M \to B$ be the associated bundle with fiber F, i.e., $M = P \times_G F$. Given a Riemannian metric g_B on B, a G-invariant Riemannian metric g_F on F and a principal connection θ on P, there exists one and only one Riemannian metric g on M such that π is a Riemannian submersion from (M, g) to (B, g_B) with totally geodesic fibers and isometric to (F, g_F) and complete horizontal distribution associated with θ. Moreover, if (B, g_B) and (F, g_F) are complete, then (M, g) is complete.*

The necessary conditions for constructing a gradient Ricci soliton $(\pi, g, \tilde{\varphi})$ with totally geodesic fibers are given by the following equations

$$^{B}\mathrm{Ric}(X, Y) - 2(A_X, A_Y) + H^{\varphi}(X, Y) = \lambda g_B(X, Y) \tag{1.2}$$

$$\mathrm{Ric}_{F_b}(U, V) + (AU, AV) = \lambda g_{F_b}(U, V) \tag{1.3}$$

$$g((\check{\delta} A)X, U) = -(A_X U)\tilde{\varphi} \tag{1.4}$$

Indeed, they follow immediately by combining the submersions equations with the Ricci soliton Eq. (1).

In [12, Proposition 3.1] Petersen and Wylie proved that a gradient Ricci soliton $\mathrm{Ric} + \nabla^2 \Psi = \lambda g$ which is an Einstein manifold, either has $\nabla^2 \Psi = 0$ or it is the Gaussian soliton. We prove that the latter case cannot occur in the class submersions with totally geodesic fibers.

Proposition 1.3 *Let $\pi : (M, g) \to (B, g_B)$ be a Riemannian submersion with totally geodesic fibers, and let φ be a smooth function on B. Suppose $(\pi, \tilde{\varphi}, g)$ is a gradient Ricci soliton. Then (M, g) is an Einstein manifold if and only if $\nabla^2\tilde{\varphi} = 0$.*

Proof: Let $(\pi, g, \tilde{\varphi})$ be with totally geodesic fibers. Suppose (M, g) is an Einstein manifold with Ricci tensor satisfying $\mathrm{Ric} = \eta g$, for some smooth function η on B. Since $T = 0$, by [1, Proposition 9.61] the following equations hold

$$^{B}\mathrm{Ric}(X, Y) - 2(A_X, A_Y) = \eta\, {}^{B}g(X, Y), \tag{1.5}$$

$$\mathrm{Ric}_F(U, V) + (AU, AV) = \eta\, g_F(U, V), \tag{1.6}$$

$$\check{\delta} A = 0. \tag{1.7}$$

On the other hand, by comparing (1.3) and (1.6), we find $\eta = \lambda$. So, from (1.5) and (1.2) we obtain $H^{\varphi} = 0$. Since $\tilde{\varphi}$ is constant along each fiber, we have $\nabla^2\tilde{\varphi}(U, V) = 0$. By the Yang-Mills condition (1.7), equation (1.4) yields $\nabla^2\tilde{\varphi}(X, U) = 0$. This proves that $\nabla^2\tilde{\varphi} = 0$. Conversely, suppose $\nabla^2\tilde{\varphi} = 0$. Then $H^{\varphi} = 0$ and $(A_X U)\tilde{\varphi} = 0$, so substituting this into (1.4), (1.3) and (1.2) we obtain that (M, g) is an Einstein manifold with Einstein constant λ. □

1.3 Construction of Gradient Ricci Soliton Riemannian Submersions

In this section, we prove Theorem 1. We begin with three basic lemmas that are straightforward generalizations of warped product case. For their proofs see Section 1.4. For our purposes we consider only Riemannian submersions $\pi : (M, g) \to (B, g_B)$ with totally geodesic fibers and integrable horizontal distribution. In this case, since A and T vanish identically, the total space (M, g) is at least locally a Riemannian product $(B \times F, g_B + g_F)$, and π is one of the canonical projections on the factors B or F.

In the spirit of the construction of Bishop and O'Neill [14], if $f > 0$ is a smooth function on B, then we can warp the metric of M. Indeed, for $p \in M$, we define the warped metric $\bar{g}$ on M, as follows

$$\bar{g}_p(E_p, G_p) = g_p(\mathcal{H}(E_p), \mathcal{H}(G_p)) + (f \circ \pi(p))^2 g_p(\mathcal{H}(E_p), \mathcal{V}(G_p)). \tag{1.8}$$

In particular, if X and Y are basic vector fields which are π-related to X and Y, respectively, and if U and V are vertical vector fields tangent to the fiber F_b with $b = \pi(p)$, then we have

$$\bar{g}(X,Y) = g_B(\bar{X},\bar{Y}), \quad \bar{g}(U,V) = (f \circ \pi(p))^2 g_{F_b}(U,V) \quad \text{and} \quad \bar{g}(X,U) = 0.$$

1.3.1 Basic Lemmas

In what follows, we will compute the geometrical features of $\bar{g}$ that will be designated by "$-$". For the proof of the first two basic lemmas it will be useful to recall the Koszul formula

$$\begin{aligned} 2\bar{g}(\bar{D}_E F, G) &= E(\bar{g}(F,G)) + F(\bar{g}(E,G)) - G(\bar{g}(E,F)) \\ &\quad + \bar{g}([E,F],G) - \bar{g}([E,G],F) - \bar{g}([F,G],E). \end{aligned} \tag{1.9}$$

Lemma 1.1 *Suppose X and Y are basic vector fields which are π-related to X and Y, respectively, and suppose U and V are vertical vector fields. Then*

1. *$\bar{D}_X Y$ is the horizontal lift of ${}^B D_{\bar{X}} \bar{Y}$;*
2. *$\bar{D}_X U = \frac{X(\tilde{f})}{\tilde{f}} U + D_X U \in \mathcal{V}$ and $\mathcal{H}(\bar{D}_X U) = 0$;*
3. *$\bar{D}_U X = \frac{X(\tilde{f})}{\tilde{f}} U \in \mathcal{V}$ and $\mathcal{H}(\bar{D}_U X) = 0$;*
4. *$\mathcal{H}(\bar{D}_U V) = -\frac{\bar{g}(U,V)}{\tilde{f}} \bar{\nabla} \tilde{f}$ and $\mathcal{V}(\bar{D}_U V) = {}^{F_b} D_U V$.*

Once we have in possession of the Riemannian connection of the warped metric $\bar{g}$, we obtain its Riemannian curvature tensor.

Lemma 1.2 *Suppose X, Y, Z are basic vector fields which are π-related to $\bar{X}, \bar{Y}, \bar{Z}$, respectively, and suppose U, V, W are vertical vector fields. Then*

1. *$\bar{R}(X,Y)Z$ is the horizontal lift of $R_B(\bar{X},\bar{Y})\bar{Z}$;*
2. *$\bar{R}(U,X)Y = -\frac{H^f(X,Y)}{f} U \in \mathcal{V}$;*
3. *$\bar{R}(X,Y)U = \frac{X(\tilde{f})}{\tilde{f}} D_Y U - \frac{Y(\tilde{f})}{\tilde{f}} D_X U + R(X,Y)U \in \mathcal{V}$;*
4. *$\bar{R}(U,V)X = 0$;*

5. $\bar{R}(X,U)V = -\frac{\bar{g}(U,V)}{\tilde{f}}\bar{D}_X\nabla\tilde{f} \in \mathcal{H}$;

6. $\bar{R}(U,V)W = R_{F_b}(U,V)W + \frac{|\nabla\tilde{f}|^2}{\tilde{f}^2}\left[\bar{g}(U,W)V - \bar{g}(V,W)U\right] \in \mathcal{V}$.

Next, by direct computations we readily get the Ricci curvature of $\bar{g}$.

Lemma 1.3 *Suppose X and Y are basic vector fields which are π-related to X and Y, respectively, and suppose U and V are vertical vector fields. Then*

1. $\overline{\mathrm{Ric}}(X,Y) = {}^B\mathrm{Ric}(X,Y) - \frac{m}{\tilde{f}}H^f(X,Y)$;

2. $\overline{\mathrm{Ric}}(X,U) = 0$;

3. $\overline{\mathrm{Ric}}(U,V) = \mathrm{Ric}_{F_b}(U,V) - \left[\tilde{f}\tilde{\Delta}f + (m-1)|\nabla\tilde{f}|^2\right]g_{F_b}(U,V)$,

where $\tilde{\Delta}f = (\Delta f)\circ\pi$.

1.3.2 Proof of the Main Theorem

As pointed out in the introduction, the first tool to prove our main theorem is a construction of gradient Ricci soliton warped metrics due to Feitosa et al. [5]. They considered a Riemannian manifold (B, g_B) endowed with two smooth functions $f > 0$ and φ satisfying (2), and they showed that f and φ must satisfy (3).

The second tool needed to prove our theorem is the generalization of warped products to bundles by Bishop and O'Neill [14] in which they considered two Riemannian manifolds (B, g_B) and (F, g_F), and a group homomorphism $h : \pi_1(B) \to I(F)$ that gave rise to a fiber bundle structure $F \longrightarrow M \overset{\pi}{\longrightarrow} B$. The Bishop and O'Neill's construction can be briefly summarized in two steps. The first step is to identify the fundamental group $\pi_1(B)$ with the deck transformation group of the simply connected covering $\beta : \tilde{B} \to B$, and let $\pi_1(B)$ act on the Riemannian product $\tilde{B}\times F$ as follows:

$$\begin{array}{ccc} \pi_1(B)\times(\tilde{B}\times F) & \longrightarrow & \tilde{B}\times F \\ (\delta,(b,p)) & \longmapsto & (\delta(b), h(\delta)(p)). \end{array}$$

This action is free, properly discontinuous and it acts by isometries, so that the quotient manifold $M = \tilde{B}\times_{\pi_1(B)} F$ has a unique Riemannian structure with the standard quotient metric g making the natural map $\nu : \tilde{B}\times F \to M$ a Riemannian covering. The second step is to obtain the projection $\pi : M \to B$ induced by the map $\tilde{B} \times F \to B$ which is given by $(b, p) \mapsto \beta(b)$. Note that if $U \subset B$ is evenly covered by β, then for each lift $\tilde{U} \subset \tilde{B}$ the map ν gives a fiber-preserving isometry of $\tilde{U} \times F$ onto $\pi^{-1}(U)$, so $\pi^{-1}(U)$ is identified with the Riemannian product $U \times F$ which is unique up to isometries of F. It follows that π is a fiber bundle with flat connection (integrable horizontal distribution) and typical fiber F invariant under the structural group. Now, we can apply a result by Vilms [8, Theorem 3.6] to conclude that π is a nontrivial totally geodesic Riemannian submersion. Moreover, the fibers are totally geodesic, see [8, Theorem 3.3].

Now, we are in a position to prove the main result of this chapter.

Theorem 1.1 *Let (B, g_B) be a complete Riemannian manifold with two smooth functions f and φ satisfying Eq.* (2)*, for any $\lambda \in \mathbb{R}$. Take the constant μ given by Eq.* (3) *and a complete Riemannian manifold (F, g_F) of dimension m and Ricci tensor $\mathrm{Ric}_F = \mu g_F$. Then, we can construct a gradient Ricci soliton $(\pi, \bar{g}, \tilde{\varphi})$ with total space $\tilde{B} \times_{\pi_1(B)} F$ having totally umbilical fibers and integrable horizontal distribution, where $\bar{g}$ is a warped metric which is obtained from the standard quotient metric g.*

Proof: Let (B, g_B) be a Riemannian manifold with two smooth functions $f > 0$ and φ satisfying (2). Take the constant μ given by (3) and a complete Riemannian manifold (F, g_F) of dimension m and Ricci tensor satisfying $\mathrm{Ric}_F = \mu g_F$. The construction made in [14] gives us the fiber bundle $F \longrightarrow M \xrightarrow{\pi} B$ with totally geodesic fibers, integrable horizontal distribution, typical fiber F, structural group $\pi_1(B)$ and total space $(\tilde{B} \times_{\pi_1(B)} F, g)$, where g is the standard quotient metric. By use of the function f, we can further warp g to obtain the metric $\bar{g}$ given by (1.8). We point out that $\pi : (M, \bar{g}) \to (B, g_B)$ became a Riemannian submersion with totally umbilical fibers. In fact, it follows immediately from part (4) of Lemma 1.1. Thus, we verify that $(\pi, \bar{g}, \tilde{\varphi})$ is a gradient Ricci soliton. Indeed, we observe that it follows from $H^{\varphi} = \nabla^2 \tilde{\varphi}$ and $H^f = \nabla^2 \tilde{f}$ along horizontal vector fields, part (1) of Lemma 1.3 and the first equation of (2), that the Ricci soliton equation (1) is satisfied on the horizontal distribution. For $X \in \mathcal{H}$ and $U \in \mathcal{V}$, we use that $\bar{\nabla} \tilde{\varphi} \in \mathcal{H}$ and part (1) of Lemma 1.1 to obtain $\bar{\nabla}^2 \tilde{\varphi}(X, U) = 0$. So, by part (2) of Lemma 1.3, the Ricci soliton equation is trivially satisfied. To conclude the proof, we take $U, V \in \mathcal{V}$, so that by definition of μ and part

(3) of Lemma 1.3 we get

$$\overline{\mathrm{Ric}}(U,V) = \left(\lambda - \frac{1}{f}\nabla\varphi(f)\right) g(U,V).$$

On the other hand, from part (4) of Lemma 1.1 we obtain

$$\bar{\nabla}^2\tilde{\varphi}(U,V) = \frac{1}{f}\nabla\varphi(f)g(U,V).$$

Combining the latter two equations we conclude that the Ricci soliton equation is again satisfied. This completes the proof of the theorem. □

1.4 Proof of the Technical Lemmas

1.4.1 Proof of Lemma 1.1

Proof: Part (1) is immediate from (1.9) and integrability of $\mathcal{H}$. To prove (2), again from (1.9) we have $2\bar{g}(\bar{D}_X U, Z) = 0$, so $\bar{D}_X U$ is vertical. Moreover,

$$\begin{aligned} 2\bar{g}(\bar{D}_X U, V) &= X(\tilde{f}^2 g(U,V)) + \tilde{f}^2\left(g([X,U],V) - g([X,V],U)\right) \\ &= 2\tilde{f}X(\tilde{f})g(U,V) + \tilde{f}^2\big[X(g(U,V)) + g([X,U],V) - g([X,V],U)\big] \\ &= 2\left(\frac{X(\tilde{f})}{\tilde{f}}\bar{g}(U,V) + \bar{g}(D_X U, V)\right) \end{aligned}$$

where in the second equality we have used (1.9) for the metric g. Next, since $\bar{g}(\bar{D}_U X, Y) = 0$, we have that $\bar{D}_U V$ is vertical. Furthermore,

$$\bar{g}(\bar{D}_U X, V) = 2\tilde{f}X(\tilde{f})g(U,V) + \tilde{f}^2\big[X(g(U,V)) + g([U,X],V) - g([X,V],U)\big].$$

Similarly, by applying (1.9) for the metric g, we obtain $\bar{D}_U X = \frac{X(\tilde{f})}{\tilde{f}}U$. To prove (4), notice that

$$2\bar{g}(\bar{D}_U V, X) = -2\tilde{f}X(\tilde{f})g(U,V) + \tilde{f}^2\big[X(g(U,V)) + g([U,X],V) - g([V,X],U)\big]. \tag{1.10}$$

On the other hand, we have

$$2g({}^{F_b}D_U V, X) = X(g(U,V)) - g([U,X],V) - g([V,X],U).$$

Since ${}^{F_b}D_U V = D_U V \in \mathcal{V}$, substituting this into (1.10) give us $\mathcal{H}(\bar{D}_U V) = -\frac{\bar{g}(U,V)}{\tilde{f}}\nabla\tilde{f}$. Moreover,

$$\begin{aligned}
2\bar{g}(\bar{D}_U V, W) &= f^2\big[U(g(V,W)) + V(g(U,W)) - W(g(U,V)) \\
&\quad + g([U,V],W) - g([U,W],V) - g([V,W],U)\big] \\
&= 2\bar{g}(D_U V, W),
\end{aligned}$$

which means that $\mathcal{V}(\bar{D}_U V) = {}^{F_b}D_U V$. This completes the proof. □

1.4.2 Proof of Lemma 1.2

Proof: We start from (6). From part (4) of Lemma 1.1, we have

$$\begin{aligned}
\bar{R}(U,V)W &= \bar{D}_U {}^{F_b}D_V W - D_U\left(\frac{\bar{g}(V,W)}{\tilde{f}}\nabla\tilde{f}\right) - \bar{D}_V({}^{F_b}D_U W) \\
&\quad + \bar{D}_V\left(\frac{\bar{g}(U,W)}{\tilde{f}}\nabla\tilde{f}\right) - {}^{F_b}D_{[U,V]}W + \frac{\bar{g}([U,V],W)}{\tilde{f}}\nabla\tilde{f} \\
&= \mathcal{V}(\bar{D}_U {}^{F_b}D_V W) + \mathcal{H}(\bar{D}_U {}^{F_b}D_V W) - \frac{1}{\tilde{f}}U(\bar{g}(V,W))\bar{\nabla}\tilde{f} \\
&\quad - \mathcal{V}(\bar{D}_V {}^{F_b}D_U W) - \mathcal{H}(\bar{D}_V {}^{F_b}D_U W) + \frac{1}{\tilde{f}}\bar{g}(U,W)\nabla\tilde{f} \\
&\quad - \frac{1}{\tilde{f}}\bar{g}(V,W)\bar{D}_U\nabla\tilde{f} - {}^{F_b}D_{[U,V]}W - \frac{\bar{g}([U,V],W)}{\tilde{f}}\nabla\tilde{f} \\
&\quad + \frac{1}{\tilde{f}}\bar{g}(U,W)\bar{D}_V\nabla\tilde{f} \\
&= R_{F_b}(U,V)W + \frac{|\nabla\tilde{f}|^2}{\tilde{f}^2}\Big[\bar{g}(U,W)V - \bar{g}(V,W)U\Big].
\end{aligned}$$

To prove part (2), we use parts (1) and (3) of Lemma 1.1, to obtain

$$\begin{aligned}
\bar{R}(U,X)Y &= \frac{(\bar{D}_X Y)(f)}{f}U - \bar{D}_X\left(\frac{X(\tilde{f})}{\tilde{f}}U\right) - \frac{Y(\tilde{f})}{\tilde{f}}[U,X] \\
&= \frac{(\bar{D}_X Y)(\tilde{f})}{\tilde{f}}U - \frac{1}{\tilde{f}}X(Y(\tilde{f}))U = -\frac{H^f(X,Y)}{\tilde{f}}U.
\end{aligned}$$

Next, from part (2) of Lemma 1.1, the following hold

$$\begin{aligned}
\bar{R}(X,Y)U &= \bar{D}_X\left(\frac{Y(\tilde{f})}{\tilde{f}}U + D_Y U\right) \\
&\quad - \bar{D}_Y\left(\frac{X(\tilde{f})}{\tilde{f}}U + D_X U\right) - \frac{[X,Y]\tilde{f}}{\tilde{f}}U - D_{[X,Y]}U
\end{aligned}$$

$$= \frac{X(\tilde{f})}{\tilde{f}} D_Y U - \frac{Y(\tilde{f})}{\tilde{f}} D_Y U + R(X,Y)U,$$

which proves part (3). To prove part (4), notice that part (3) of Lemma 1.1 give us that

$$\begin{aligned}\bar{R}(U,V)X &= \bar{D}_U \left(\frac{X(\tilde{f})}{\tilde{f}} V \right) - \bar{D} \left(\frac{X(\tilde{f})}{\tilde{f}} U \right) - \frac{X(\tilde{f})}{\tilde{f}} [U,V] \\ &= \frac{1}{\tilde{f}} \Big[U(X(\tilde{f}))V - V(X(\tilde{f}))U \Big] + \frac{X(\tilde{f})}{\tilde{f}} \Big[\bar{D}_U V - \bar{D}_V U - [U,V] \Big] = 0\end{aligned}$$

where in the latter equality we have used (1.9) for the metric g and the fact that $\tilde{f}$ is constant along the fibers. Part (5) follows from parts (1) and (2) of Lemma 1.1 as follows

$$\begin{aligned}\bar{R}(X,U)V &= \bar{D}_X \big(\mathcal{H}(\bar{D}_U V) + \mathcal{V}(\bar{D}_U V) \big) - \bar{D}_U \left(\frac{X(\tilde{f})}{\tilde{f}} V + D_X V \right) \\ &\quad - \mathcal{H}(\bar{D}_{[X,U]} V) - \mathcal{V}(\bar{D}_{X,U} V) \\ &= \bar{D}_X \left(- \frac{\bar{g}(U,V)}{\tilde{f}} \nabla \tilde{f} \right) + \bar{D}_X {}^{F_b} D_U V - \bar{D}_U \left(\frac{X(\tilde{f})}{\tilde{f}} V \right) - \bar{D}_U D_X V \\ &\quad + \frac{\bar{g}([X,U],V)}{\tilde{f}} \nabla \tilde{f} - {}^{F_b} D_{[X,U]} V \\ &= - \frac{\bar{g}(U,V)}{\tilde{f}} \bar{D}_X \nabla \tilde{f}\end{aligned}$$

where in the latter equality we have used the fact that $R(X,U)V = 0$ when the fibers are totally geodesic submanifolds of the total space and the horizontal distribution is integrable. Moreover, part (1) follows immediately from part (1) of Lemma 1.1, which concludes this proof. □

1.4.3 Proof of Lemma 1.3

Proof: We take $p \in M$, and orthonormal basis $\{X_i\}$ and U_j of the distributions $\mathcal{H}_p$ and $\mathcal{V}_p$, respectively. Taking traces of equations (1) and (2) of Lemma 1.2, we have

$$\bar{\text{Ric}}(X,Y) = {}^B\text{Ric}(X,Y) - \frac{m}{\tilde{f}} H^f(X,Y).$$

This latter proves part (1). Next, to prove part (2), we take trace of parts (5) and (6) of Lemma 1.2, to obtain

$$\bar{\text{Ric}}(U,V) = \sum_{i=1}^{n} \bar{g} \left(- \frac{\bar{g}(U,V)}{\tilde{f}} \bar{D}_{X_i} \nabla \tilde{f}, X_i \right) + \sum_{j=1}^{m} \bar{g}(\bar{R}_{F_b}(U_j,U)V, U_j)$$

$$
+\frac{|\nabla \tilde{f}|^2}{\tilde{f}} \sum_{j=1}^{m} [\bar{g}(U_j, V)\bar{g}(U, U_j) - \bar{g}(U_j, U_j)\bar{g}(U, V)]
$$

$$
= \mathrm{Ric}_{F_b}(U, V) - \left[\tilde{f}\tilde{\Delta}\tilde{f}(m-1)|\nabla \tilde{f}|^2\right] g_{F_b}(U, V).
$$

To prove part (3), by using parts (3) and (4) of Lemma 1.2, we easily verify that $\overline{\mathrm{Ric}}(X, U) = 0$, so we conclude the proof. □

Chapter 2

Gradient Ricci Soliton Warped Products

In this chapter we prove triviality and nonexistence results for gradient Ricci solitons that are realized as warped products. We start this chapter by studying the Ricci Hessian type manifolds as the main tool to prove our theorems.

2.1 Ricci-Hessian type manifolds

Let $(B^n, e^{-h}d\text{vol})$ be an n-dimensional weighted Riemannian manifold, h be a smooth function on B and $d$vol be the Riemannian volume density on (B^n, g_B). A natural extension of the Ricci tensor to weighted manifolds is the *m-Bakry-Emery Ricci* tensor

$$\text{Ric}_h^m = \text{Ric} + \nabla^2 h - \frac{1}{m} dh \otimes dh, \quad \text{for some } 0 < m \leq +\infty.$$

We consider (B^n, g_B) endowed with two smooth functions φ and $f > 0$. We can further consider the weighted Riemannian manifold $(B^n, e^{-\psi}d\text{vol})$, where ψ is the smooth function on B given by $\psi = \varphi - m \ln f$, for $0 < m < +\infty$. For our purposes, we will work with the following modification of the Ricci tensor:

$$\text{Ric}_{\varphi,f}^m = \text{Ric} + \nabla^2 \varphi - \frac{m}{f} \nabla^2 f. \tag{2.1}$$

This tensor can also be viewed in the following way:

$$\text{Ric}_{\varphi,f}^m = \text{Ric} + \nabla^2 \varphi + \nabla^2 \xi - \frac{1}{m} d\xi \otimes d\xi,$$

where $\xi = -m \ln f$. So, when φ is constant we recover the m-Bakry-Emery Ricci tensor Ric_ξ^m. An interesting case occurs when the metric g_B satisfies

$$\mathrm{Ric}_{\varphi,f}^m = \lambda g_B, \tag{2.2}$$

for some smooth function λ on B. A motivation to study this equation comes from Maschler's work [6] in which he was interested in conformal changes of Kähler-Ricci solitons in order to obtain new Kähler-metrics. To do this, he introduced the notion of *Ricci-Hessian equation*, namely

$$\mathrm{Ric} + \alpha \nabla^2 \psi = \beta g, \tag{2.3}$$

where α and β are smooth functions. Feitosa et al. [5, Remark 1] showed how Eq. (2.2) can be reduced to a Ricci-Hessian equation. Example 1 in [4] shows that the standard sphere and the hyperbolic space both satisfy Eq. (2.2).

We recapitulate from Chapter 1 that another motivation to study Eq. (2.2) is that the base spaces of gradient Ricci soliton warped products satisfy the referred equation for a positive integer m and a constant λ. To see this, we briefly describe the construction by Feitosa et al. [4]. They considered a complete Riemannian manifold (B^n, g_B) with two smooth functions $f > 0$ and φ satisfying

$$\mathrm{Ric} + \nabla^2 \varphi - \frac{m}{f} \nabla^2 f = \lambda g_B \tag{2.4}$$

and

$$2\lambda\varphi + |\nabla\varphi|^2 + \Delta\varphi + \frac{m}{f} \nabla\varphi(f) = c, \tag{2.5}$$

for some constants $\lambda, m, c \in \mathbb{R}$, with $m \neq 0$. By [4, Proposition 3], the functions f and φ satisfy

$$\lambda f^2 + f\Delta f + (m-1)|\nabla f|^2 - f\nabla\varphi(f) = \mu, \tag{2.6}$$

for some constant $\mu \in \mathbb{R}$. In summary, we will need the following result (see Proposition 2 and Theorem 3 in [4]).

Proposition 2.1 (Feitosa et al. [4]) *Let $M = B^n \times_f F^m$ be a gradient Ricci soliton warped product with potential function $\tilde{\varphi}$. Then, Eqs. (2.4) and (2.5) hold on B and the fiber F is an Einstein manifold with Ricci tensor $\mathrm{Ric}_F = \mu g_F$, where μ is given by (2.6). Conversely, let B be a complete Riemannian manifold with two smooth functions φ and*

$f > 0$ satisfying (2.4) *and* (2.5)*, for any $\lambda \in \mathbb{R}$. Take the constant μ given by* (2.6) *and a complete Riemannian manifold F of dimension m and Ricci tensor* $\mathrm{Ric}_F = \mu g_F$*. Then, $(B^n \times_f F^m, g, \tilde{\varphi})$ is a gradient Ricci soliton warped product.*

We point out that the potential function used in Proposition 2.1 is justified by the next result.

Lemma 2.1 (Borges and Tenenblat [16]) *Let $M = B^n \times_f F^m$ be a warped product with metric $g = g_B + f^2 g_F$, where $f > 0$ is a smooth function on B. If (M, g) is a gradient Ricci soliton, then the potential function depends only on the base.*

Proof: The proof follows the same argument as the more general case in [16]. Suppose (M, g) is a gradient Ricci soliton warped product with potential function Ψ on M. Then, the metric g satisfies

$$\mathrm{Ric} + \nabla^2 \Psi = \lambda g, \tag{2.7}$$

for some constant $\lambda \in \mathbb{R}$. By the standard expressions for the Ricci curvature and for the Hessian of such a function Ψ on M (see Bishop and O'Neill [14]), we obtain

$$0 = (\nabla^2 \Psi)(X, U) = X(U(\Psi)) - (\nabla_X U)(\Psi) = X(U(\Psi)) - \frac{X(f)}{f} U(\Psi),$$

for all X, U horizontal and vertical vector fields, respectively. Next, we compute

$$X(U(\Psi f^{-1})) = X(U(\Psi) f^{-1}) = \frac{1}{f} \left[X(U(\Psi)) - \frac{X(f)}{f} U(\Psi) \right] = 0.$$

Hence, the function $U(\Psi f^{-1})$ depends only on the fiber F. Thus, without loss of generality, we can assume that $\Psi = \varphi + f\phi$, for some functions φ on B and ϕ on F. Now we take a unitary geodesic γ on B, so that equation (2.7) reads along γ as

$$\mathrm{Ric}_B(\gamma', \gamma') + \varphi'' - \frac{m}{f} f'' + \phi f'' = \lambda.$$

Thus, for any vertical vector field V, one has $f'' V(\phi) = 0$. Since we are considering f positive and nonconstant, there exists a point $(p, q) \in M$ such that $f''(p) \neq 0$, for all $q \in F$. Then, ϕ must be constant on F, i.e., Ψ depends only on B. □

We now define precisely the main object of study in this chapter. A *Ricci-Hessian type manifold* is a complete Riemannian manifold (B^n, g_B) endowed with two smooth functions $f > 0$ and φ satisfying Eq. (2.2) for a positive integer m and a constant

λ. For simplicity, we will say that (B^n, g_B, ψ) is a Ricci-Hessian type manifold, where $\psi = \varphi - m \ln f$. Moreover, we will use the weighted Riemannian volume density $e^{-\psi} d\mathrm{vol}$ on (B^n, g_B, ψ) whenever it is needed.

With these notations in mind we prove now the following formulas, which are derived in the same way as in Petersen and Wylie [12, Lemma 2.5].

Lemma 2.2 *Let (B^n, g_B, ψ) be a Ricci-Hessian type manifold with scalar curvature S. Then, the following hold:*

1. $\mathring{\mathrm{Ric}} = -\mathring{\nabla}^2 \varphi + \frac{m}{f} \mathring{\nabla}^2 f$.
2. $\frac{1}{2} dS = \mathrm{Ric}(\nabla \psi, \cdot) - \frac{m}{f^2}(\Delta f df - \frac{1}{2} d|\nabla f|^2)$.
3. $R(X,Y)(\nabla \psi) = (\nabla_Y \mathrm{Ric})(X) - (\nabla_X \mathrm{Ric})(Y) - \frac{m}{f^2}(df \wedge \nabla^2 f)(X,Y)$, *for all vector fields X, Y in $\mathfrak{X}(B)$.*
4. $\frac{1}{2} \Delta_\psi S = \frac{S}{n}(\lambda n - S) - |\mathring{\mathrm{Ric}}|^2 + m\,(2\,\mathrm{Ric}(\nabla \ln f, \nabla \ln f) + |\nabla^2 \ln f|^2 - (\Delta \ln f)^2)$.

Proof: From Eq. (2.2), we have

$$\mathring{\mathrm{Ric}} := \mathrm{Ric} - \frac{S}{n} g = -\nabla^2 \varphi + \left(\lambda - \frac{S}{n}\right) g + \frac{m}{f} \nabla^2 f.$$

Taking trace of Eq. (2.2) we also get

$$S + \Delta \varphi = n\lambda + \frac{m}{f} \Delta f.$$

Combining the latter two equations we obtain part (1). For part (2) recall that for any smooth function v on B the following holds

$$\mathrm{div}\,\mathrm{Ric} = \frac{1}{2} dS \quad \text{and} \quad \Delta dv = d\Delta v + \mathrm{Ric}(\nabla v, \cdot),$$

Hence, by Eq. (2.2) we obtain

$$\begin{aligned}
\frac{dS}{2} &= \mathrm{div}(\lambda g - \nabla^2 \varphi + \frac{m}{f} \nabla^2 f) \\
&= -d\Delta \varphi - \mathrm{Ric}(\nabla \varphi, \cdot) + \frac{m}{f}(d\Delta f + \mathrm{Ric}(\nabla f, \cdot)) + \nabla^2 f(\nabla(\frac{m}{f}), \cdot) \\
&= \frac{m}{f^2} \Delta f df + dS - \mathrm{Ric}(\nabla(\varphi - m \ln f), \cdot) - \frac{m}{f^2} \nabla^2 f(\nabla f, \cdot).
\end{aligned}$$

Since $d|\nabla f|^2 = 2(\nabla^2 f)(\nabla f)$, then we get part (2).

Next, recall that for any smooth function v on B and X, Y in $\mathfrak{X}(B)$, the Riemann curvature tensor satisfies

$$R(X,Y)\nabla v = (\nabla_X \nabla^2 v)(Y) - (\nabla_Y \nabla^2 v)(X). \tag{2.8}$$

We take $v = \varphi$ and by using of Eq. (2.2) again, we have

$$\begin{aligned}
(\nabla_X \nabla^2 \varphi)(Y) &= \left(\nabla_X \left(\lambda I + \frac{m}{f}\nabla^2 f - \operatorname{Ric}\right)\right)(Y) \\
&= -(\nabla_X \operatorname{Ric})(Y) + \left(\nabla_X \left(\frac{m}{f}\nabla^2 f\right)\right)(Y) \\
&= -(\nabla_X \operatorname{Ric})(Y) + \frac{m}{f}(\nabla_X \nabla^2 f)(Y) - \frac{m}{f^2}(df \otimes \nabla^2 f)(X,Y).
\end{aligned}$$

Combining the previous equality with (2.8) we get

$$R(X,Y)\nabla\varphi = (\nabla_Y \operatorname{Ric})(X) - (\nabla_X \operatorname{Ric})(Y) + \frac{m}{f}R(X,Y)\nabla f - \frac{m}{f^2}(df \wedge \nabla^2 f)(X,Y).$$

This proves part (3). Moreover, by using part (2), we get

$$\begin{aligned}
(\nabla_X \operatorname{Ric})(\nabla\psi) &= \nabla_X \left(\operatorname{Ric}(\nabla\psi)\right) - \operatorname{Ric}\left(\nabla_X \left(\nabla\varphi - \frac{m}{f}\nabla f\right)\right) \\
&= \nabla_X \left(\frac{1}{2}\nabla S + \frac{m}{f^2}\left(\Delta f \nabla f - \frac{1}{2}\nabla|\nabla f|^2\right)\right) \\
&\quad - \operatorname{Ric}\left(\left(\nabla^2\varphi - \frac{m}{f}\nabla^2 f\right)(X) + \frac{m}{f^2}X(f)\nabla f\right) \\
&= \frac{1}{2}\nabla^2 S(X) + \nabla_X \left(\frac{m}{f^2}\left(\Delta f \nabla f - \frac{1}{2}\nabla|\nabla f|^2\right)\right) \\
&\quad - (\operatorname{Ric}\circ(\lambda I - \operatorname{Ric}))(X) - \frac{m}{f^2}X(f)\operatorname{Ric}(\nabla f).
\end{aligned}$$

Substituting this into part (3) we have

$$\begin{aligned}
R(X,\nabla\psi)\nabla\psi &= (\nabla_{\nabla\psi} \operatorname{Ric})(X) - \frac{1}{2}\nabla^2 S(X) - \nabla_X \left(\frac{m}{f^2}\left(\Delta f \nabla f - \frac{1}{2}\nabla|\nabla f|^2\right)\right) \\
&\quad + (\operatorname{Ric}\circ(\lambda I - \operatorname{Ric}))(X) + \frac{m}{f^2}X(f)\operatorname{Ric}(\nabla f) - \frac{m}{f^2}(df \wedge \nabla^2 f)(X,\nabla\psi).
\end{aligned}$$

Taking the trace of the latter equation we obtain

$$\begin{aligned}
\frac{1}{2}\Delta_\psi S &= \operatorname{tr}\left(\operatorname{Ric}\circ(\lambda I - \operatorname{Ric})\right) + \frac{m}{f^2}\operatorname{Ric}(\nabla f, \nabla f) \\
&\quad - \operatorname{div}\left(\frac{m}{f^2}\left(\Delta f \nabla f - \frac{1}{2}\nabla|\nabla f|^2\right)\right).
\end{aligned} \tag{2.9}$$

For any smooth function $v > 0$ on B it is true that

$$\nabla^2(\ln v) = \frac{1}{v}\nabla^2 v - \frac{1}{v^2} dv \otimes dv. \tag{2.10}$$

So,

$$\begin{aligned}\frac{m}{f^2}\left(\Delta f \nabla f - \frac{1}{2}\nabla|\nabla f|^2\right) &= m\left(\frac{\Delta f}{f}\nabla(\ln f) - \frac{1}{2f^2}\nabla(f^2|\nabla(\ln f)|^2)\right) \\ &= \frac{m}{f}\Delta f \nabla(\ln f) - \frac{m}{f}|\nabla(\ln f)|^2\nabla f - \frac{m}{2}\nabla|\nabla(\ln f)|^2 \\ &= m\Delta(\ln f)\nabla(\ln f) - \frac{m}{2}\nabla|\nabla(\ln f)|^2.\end{aligned}$$

Whence,

$$\begin{aligned}\operatorname{div}\left(\frac{m}{f^2}\left(\Delta f \nabla f - \frac{1}{2}\nabla|\nabla f|^2\right)\right) =& m(\Delta(\ln f))^2 + m\langle\nabla(\ln f), \nabla\Delta(\ln f)\rangle \\ & - \frac{m}{2}\Delta|\nabla(\ln f)|^2.\end{aligned}$$

By (2.9), we get

$$\begin{aligned}\frac{1}{2}\Delta_\psi S = \operatorname{tr}(\operatorname{Ric}\circ(\lambda I - \operatorname{Ric})) + \frac{m}{f^2}\operatorname{Ric}(\nabla f, \nabla f) - m(\Delta(\ln f))^2 \\ - m\,\langle\nabla(\ln f), \nabla\Delta(\ln f)\rangle + \frac{m}{2}\Delta|\nabla(\ln f)|^2.\end{aligned}$$

By using the Bochner formula for the function $\ln f$ we rewrite the equation above as follows

$$\frac{1}{2}\Delta_\psi S = \operatorname{tr}(\operatorname{Ric}\circ(\lambda I - \operatorname{Ric})) + m\big(2\operatorname{Ric}(\nabla \ln f, \nabla \ln f) + |\nabla^2 \ln f|^2 - (\Delta \ln f)^2\big).$$

Noting that $\operatorname{tr}(\operatorname{Ric}\circ(\lambda I - \operatorname{Ric})) = \lambda S - |\operatorname{Ric}|^2$ and $|\mathring{\operatorname{Ric}}|^2 = |\operatorname{Ric}|^2 - \frac{S^2}{n}$ we conclude the proof. □

Now we continue by fixing notation and making comments about facts that will be used henceforth. Let h be a smooth function on a Riemannian manifold (B, g_B), and let

$$\operatorname{Ric}_h = \operatorname{Ric} + \nabla^2 h \tag{2.11}$$

denote the *Bakry-Emery Ricci* tensor of the weighted Riemannian manifold $(B, e^{-h}d\mathrm{vol})$. Thus, $(B, g_B, \nabla h)$ is a gradient Ricci soliton provided the corresponding complete weighted Riemannian manifold has constant Ric_h-curvature, that is, $\operatorname{Ric}_h \equiv \lambda$, for some $\lambda \in \mathbb{R}$.

Associated to Ric_h we have the following Bochner type formula

$$\frac{1}{2}\Delta_h|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_h u\rangle + \mathrm{Ric}_h(\nabla u, \nabla u), \tag{2.12}$$

for $u \in C^\infty(B)$. Recall that the second order operator h-Laplacian $\Delta_h u = e^h \mathrm{div}(e^{-h}\nabla u)$ is formally self-adjoint in $L^2(B, e^{-h}d\mathrm{vol})$ space, and that the *weak maximum principle at infinity* for Δ_h on (B, g_B) holds if given a C^2 function $u : B \to \mathbb{R}$ satisfying $\sup_B u = u^* < +\infty$, there exists a sequence $\{x_k\} \subset B$ along which

$$(i)\ u(x_k) \geq u^* - \frac{1}{k} \quad \text{and} \quad (ii)\ (\Delta_h u)(x_k) \leq \frac{1}{k}, \quad \text{for all } k \in \mathbb{N}.$$

Since the class of Ricci-Hessian type manifolds contains both m-quasi Einstein manifolds and gradient Ricci solitons, the weak maximum principle at infinity is very useful in order to study such manifolds. In this setting, for a Ricci-Hessian type manifold (B, g_B, ψ) we take $h = \psi$ in Eq. (2.11) and $v = f$ in Eq. (2.10) to obtain the very convenient identity

$$\mathrm{Ric}_\psi = \mathrm{Ric}^m_{\varphi,f} + \frac{m}{f^2} df \otimes df. \tag{2.13}$$

We point out that Eq. (2.13) has two important immediate consequences. The first one is the following Bochner type formula for the modified Ricci tensor $\mathrm{Ric}^m_{\varphi,f}$.

Lemma 2.3 *Let (B^n, g_B, ψ) be a Ricci-Hessian type manifold. Then, for a smooth function u on B the following holds*

$$\frac{1}{2}\Delta_\psi|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_\psi u\rangle + \mathrm{Ric}^m_{\varphi,f}(\nabla u, \nabla u) + \frac{m}{f^2}\langle \nabla f, \nabla u\rangle^2. \tag{2.14}$$

Proof: It is an immediate consequence of equations (2.12) and (2.13). □

The second consequence of Eq. (2.13) is the validity of the weak maximum principle at infinity for Δ_ψ on a Ricci-Hessian type manifold (B, g_B, ψ). In view of this, we shall refer to the following two results. First, we recall a weighted-volume comparison established in [17, Theorem 4.1] as follows.

Proposition 2.2 (Wei and Wylie [17]) *Let $(B, e^{-h}d\mathrm{vol})$ be a complete weighted Riemannian manifold. Suppose that*

$$\mathrm{Ric}_h \geq \lambda,$$

for some $\lambda \in \mathbb{R}$. Then, having fixed $R_0 > 0$, there are constants $A, B, C > 0$ such that, for every $r \geq R_0$,

$$\mathrm{vol}_h(B_r) \leq A + B\int_{R_0}^r e^{-\lambda t^2 + Ct}dt.$$

The next result states the validity of a weak form of the maximum principle at infinity for the h-Laplacian, under weighted-volume growth conditions obtained in [15, Theorem 9]. In what follows, $L^1(+\infty)$ stands for the space of the integrable functions at infinity, that is, a function v belongs to $L^1(+\infty)$ if there exists $R \in \mathbb{R}$ such that $v \in L^1([R, +\infty))$.

Proposition 2.3 (Pigola et al. [15]) *Let* $(B, e^{-h}d\text{vol})$ *be a complete weighted Riemannian manifold satisfying the volume growth condition*

$$\frac{r}{\ln \text{vol}_h(B_r)} \notin L^1(+\infty). \tag{2.15}$$

Then, the weak maximum principle at infinity for the h-Laplacian holds on B.

In summary, one readily has the following proposition.

Proposition 2.4 *The weak maximum principle at infinity for the ψ-Laplacian holds on a Ricci-Hessian type manifold* (B^n, g_B, ψ).

Proof: Since $\text{Ric}^m_{\varphi,f} = \lambda$, by Eq. (2.13) we have $\text{Ric}_\psi \geq \lambda$. Moreover, this lower bound on Ric_ψ implies the volume growth condition (2.15), see [9, pg. 107]. Thus, we apply Propositions 2.2 and 2.3 to the weighted manifold $(B, e^{-\psi}d\text{vol})$ to obtain the validity of the weak maximum principle at infinity for the ψ-Laplacian on B. □

We point out that Proposition 2.4 is crucial to prove our results under L^∞ conditions. We also note that more generally this weak maximum principle at infinity holds if the weighted Riemannian manifold $(B, e^{-\psi}d\text{vol})$ satisfies $\text{Ric}^m_{\varphi,f} \geq \lambda$.

2.2 Ricci-Hessian type manifolds as base spaces of gradient Ricci soliton warped products

In this section, we elaborate the proofs of the main theorems of this chapter. First of all we restrict ourselves to Ricci-Hessian type manifolds as base spaces of gradient Ricci soliton warped products, which means that they must satisfy the additional Eq. 2.5 as well as Eq. 2.6. We start by using equations (2.4), (2.5) and (2.6) to obtain the following lemma.

Lemma 2.4 *Let (B^n, g_B, ψ) be a Ricci-Hessian type manifold satisfying the additional Eq.* (2.5)*. Then,*

$$\begin{aligned}\Delta\psi &= n\lambda - S + m|\nabla(\ln f)|^2,\\ \Delta_\psi\varphi &= c - 2\lambda\varphi,\\ \Delta_\psi(\ln f) &= \frac{1}{f^2}(\mu - \lambda f^2).\end{aligned}$$

Proof: The first equation is immediately obtained by taking the trace of Eq. (2.4). The second and third equations follow from equations (2.5) and (2.6), respectively. □

We state now the following version of a L^p-Liouville-type theorem for the h-Laplacian on a complete weighted Riemannian manifold $(B, e^{-h}d\text{vol})$ established in [15, Theorem 14]. This result will be important to prove our theorems under L^p conditions.

Proposition 2.5 (Pigola et al. [15]) *Let $(B, g_B, e^{-h}d\text{vol})$ be a complete weighted manifold. Assume that $u \in Lip_{loc}(B)$ satisfy*

$$u\Delta_h u \geq 0, \quad \textit{weakly on } (B, e^{-h}d\text{vol}).$$

If, for some $p > 1$,

$$\frac{1}{\int_{\partial B_r} |u|^p e^{-h} d\text{vol}_{n-1}} \notin L^1(+\infty), \tag{2.16}$$

then u is constant.

Remark 2.1 *By [15, Remark 15], if $u \in L^p(B, e^{-h}d\text{vol})$ then the condition* (2.16) *is satisfied. Note also, that no sign condition is required on u. Moreover, if the locally Lipschitz function u satisfies both $\Delta_h u \geq 0$ and the non-integrability* (2.16)*, then applying Proposition 2.5 to $u_+ = \max\{u, 0\}$ gives that either u is constant, or $u \leq 0$. In a similar way, we can apply Theorem 2.5 to $u_- = \max\{-u, 0\}$.*

As a consequence of Proposition 2.5 and Remark 2.1 we have a L^p-Liouville type result for Ricci-Hessian type manifolds.

Corollary 2.1 *Let (B, g_B, ψ) be a Ricci-Hessian type manifold. If $0 \leq u \in Lip_{loc}(B)$ satisfies $\Delta_\psi u \geq 0$ and $u \in L^p(B, e^{-\psi}d\text{vol})$ for some $1 < p < +\infty$, then u is constant.*

2.2.1 Results under $L^{1<p\leq+\infty}$ conditions

In what follows we will prepare the ground for the proof of the main theorems in this chapter. We star with the following proposition.

Proposition 2.6 *Let (B^n, g_B, ψ) be a Ricci-Hessian type manifold satisfying the additional Eq.* (2.5)*. Suppose $\lambda \leq 0$. Then,*

a) If $\varphi \in L^p(B, e^{-\psi}\mathrm{dvol})$, for some $1 < p < +\infty$, then either φ is constant and (B^n, g_B) is an m-quasi-Einstein manifold or φ has a sign on B.

b) For $\lambda = 0$, if $\mu \geq 0$ and $f \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$, then f is constant, $\mu = 0$ and $(B^n, g_B, \nabla\varphi)$ is a gradient steady Ricci soliton.

c) For $\lambda < 0$, there is no such a Ricci-Hessian type manifold provided that $\mu \geq 0$ and f satisfies either of the following conditions: $f \in L^\infty(B)$ or $f \in L^p(B, e^{\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.

Proof: To prove part a), suppose $\varphi \in L^p(B, e^{-\psi}\mathrm{dvol})$, for some $1 < p < +\infty$. By Lemma 2.4 we obtain that

$$\Delta_\psi \varphi = c - 2\lambda\varphi.$$

For $c \geq 0$, we consider $\varphi_+ = \max\{\varphi, 0\}$. Since $\lambda \leq 0$, we get $\Delta_\psi \varphi_+ = c - 2\lambda\varphi_+ \geq 0$. Applying Corollary 2.1 to $\varphi_+ \in L^p(B, e^{-\psi}\mathrm{dvol})$, gives us that φ_+ is constant. Hence, if there exists a point $x_0 \in B$ such that $\varphi(x_0) \geq 0$, then $\varphi \equiv \varphi(x_0) \geq 0$ and B is a m-quasi-Einstein manifold with potential function $\xi = -m \ln f$. Otherwise, we have $\varphi < 0$ on B. In a similar way, for $c \leq 0$ we apply Corollary 2.1 to $\varphi_- \in L^p(B, e^{-\psi}\mathrm{dvol})$ to obtain that either B is an m-quasi-Einstein manifold or $\varphi > 0$ on B.

Now we prove part b). By Lemma 2.4, we have

$$f\Delta_\psi f = \mu - \lambda f^2 + |\nabla f|^2. \tag{2.17}$$

For $\lambda = 0$, we suppose that $\mu \geq 0$ and $f \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$. From Eq. (2.17) we have $f\Delta_\psi f \geq 0$. Then we apply Corollary 2.1 to obtain that f is a constant. So, $(B^n, g_B, \nabla\varphi)$ is a gradient steady Ricci soliton with potential function φ and from Eq. (2.17) we conclude that $\mu = 0$.

We prove now part c). Since $\lambda < 0$ and $\mu \geq 0$, the nonexistence result for the standard Riemannian case is trivial by the third equation of Lemma 2.4. We suppose now that f is nonconstant, so that we can use Eq. (2.17) to obtain

$$\Delta_\psi f \geq -\lambda f \geq 0. \tag{2.18}$$

Assume that $f \in L^\infty(B)$, i.e., $f^* = \sup_B f < +\infty$. By the weak maximum principle at infinity for the ψ-Laplacian, there exists a sequence $\{x_k\} \subset B$ along which

$$\lim_{k\to+\infty} f(x_k) = f^* \quad \text{and} \quad \limsup_{k\to+\infty}(\Delta_\psi f)(x_k) \leq 0.$$

Thus, evaluating (2.18) along $\{x_k\}$ and taking $\limsup$ as $k \to +\infty$, we obtain that $-\lambda f^* = 0$, which is a contradiction, because $\lambda < 0$ and $f^* > 0$.

Next, we assume that $f \in L^p(B, e^{-\psi}d\mathrm{vol})$ for some $1 < p < +\infty$. Since $\lambda < 0$, we obtain that $\Delta_\psi f \geq 0$. So, we apply Corollary 2.1 to conclude that f must be constant, which is again a contradiction. □

We now combine Lemmas 2.3 and 2.4 to obtain the following Bochner type formulas for Ricci-Hessian type manifolds.

Lemma 2.5 *Let (B^n, g_B, ψ) be a Ricci-Hessian type manifold satisfying the additional Eq.* (2.5)*. Then, for a smooth function u on B the following holds:*

a) $\frac{1}{2}\Delta_\psi|\nabla\varphi|^2 = |\nabla^2\varphi|^2 - \lambda|\nabla\varphi|^2 + \frac{m}{f^2}\langle\nabla\varphi, \nabla f\rangle^2$.

b) $\frac{1}{2}\Delta_\psi|\nabla \ln f|^2 = |\nabla^2 \ln f|^2 - \frac{2\mu}{f^2}|\nabla \ln f|^2 + \lambda|\nabla \ln f|^2 + \frac{m}{f^2}\langle\nabla \ln f, \nabla f\rangle^2$.

Proof: It is immediate by taking $u = \varphi$ and $\ln f$ in Lemma 2.3, respectively, and by using the equations of Lemma 2.4. □

In addition to Propositions 2.4 and 2.5 we will also use the following Bochner type inequalities to prove our results.

Lemma 2.6 *Let (B^n, g_B, ψ) be a Ricci-Hessian type manifold satisfying the additional Eq.* (2.5)*. Then, the following inequalities hold:*

$$|\nabla\varphi|\Delta_\psi|\nabla\varphi| \geq -\lambda|\nabla\varphi|^2 + \frac{m}{f^2}\langle\nabla\varphi, \nabla f\rangle^2 \tag{2.19}$$

and

$$|\nabla \ln f|\Delta_\psi|\nabla \ln f| \geq \lambda|\nabla \ln f|^2 - \frac{2\mu}{f^2}|\nabla \ln f|^2 + \frac{m}{f^2}\langle\nabla \ln f, \nabla f\rangle^2. \tag{2.20}$$

Proof: For a smooth function u on B we have

$$\frac{1}{2}\Delta_\psi|\nabla u|^2 = |\nabla u|\Delta_\psi|\nabla u| + |\nabla|\nabla u||^2.$$

Now, combining this latter equality with the Kato inequality, namely

$$|\nabla^2 u|^2 \geq |\nabla|\nabla u||^2, \quad u \in C^\infty(B), \tag{2.21}$$

one has

$$|\nabla u|\Delta_\psi|\nabla u| \geq \frac{1}{2}\Delta_\psi|\nabla u|^2 - |\nabla|\nabla u||^2.$$

Taking $u = \varphi$ into the latter inequality and using part a) of Lemma 2.5 we have

$$|\nabla\varphi|\Delta_\psi|\nabla\varphi| = \frac{1}{2}\Delta_\psi|\nabla\varphi|^2 - |\nabla|\nabla\varphi||^2 \geq -\lambda|\nabla\varphi|^2 + \frac{m}{f^2}\langle\nabla\varphi, \nabla f\rangle^2.$$

The second required inequality is analogously obtained. □

We continue to prove some triviality and nonnexistence results for Ricci-Hessian type manifolds satisfying the additional Eq. (2.5) as follows.

Proposition 2.7 *Let (B^n, g_B, ψ) be a Ricci-Hessian type manifold with $\lambda < 0$ satisfying the additional Eq.* (2.5) *. Then, the parameter function φ is a constant provided that it satisfies either of the following conditions: $|\nabla\varphi| \in L^\infty(B)$ or $|\nabla\varphi| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$. In this case B must be an m-quasi-Einstein manifold.*

Proof: Since $\lambda < 0$, from part a) of Lemma 2.5 we have

$$\Delta_\psi|\nabla\varphi|^2 \geq -2\lambda|\nabla\varphi|^2 \geq 0. \tag{2.22}$$

Assuming that $|\nabla\varphi| \in L^\infty(B)$, by the weak maximum principle at infinity for the ψ-Laplacian there exists a sequence $\{x_k\} \subset B$ such that

$$\limsup_{k\to+\infty}\,(\Delta_\psi|\nabla\varphi|^2)(x_k) \leq 0 \quad \text{and} \quad \lim_{k\to+\infty}|\nabla\varphi|^2(x_k) = \sup_B|\nabla\varphi|^2.$$

Thus, evaluating (2.22) along $\{x_k\}$ and taking lim sup as $k \to +\infty$, we get

$$\lambda\sup_B|\nabla\varphi|^2 = 0.$$

Then, we must have $|\nabla\varphi| = 0$, that is, φ is a constant.

Assume now that $|\nabla\varphi| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$. Since $\lambda < 0$, from (2.19) we obtain that $|\nabla\varphi|\Delta_\psi|\nabla\varphi| \geq 0$. We apply Corollary 2.1 to obtain that $|\nabla\varphi|$ is a constant. Substituting this latter fact into (2.19), φ must be constant. This concludes the proof. $\square$

Next, we prove the following nonexistence result.

Proposition 2.8 *There is no Ricci-Hessian type manifold (B, g_B, ψ) satisfying the additional Eq.* (2.5) *with $\lambda > 0$ and $\mu \leq 0$ provided that the parameter function f satisfies either of the following conditions: $|\nabla \ln f| \in L^\infty(B)$ or $|\nabla \ln f| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

Proof: Since $\lambda > 0$ and $\mu \leq 0$, the constant case is trivial by the third equation of Lemma 2.4. So that, we suppose now f is nonconstant. From part b) of Lemma 2.5 we have

$$\Delta_\psi |\nabla \ln f|^2 \geq 2\lambda |\nabla \ln f|^2 \geq 0. \tag{2.23}$$

Assuming that $|\nabla \ln f| \in L^\infty(B)$, by the weak maximum principle at infinity for the ψ-Laplacian there exists a sequence $\{x_k\} \subset B$ such that

$$\limsup_{k\to+\infty} (\Delta_\psi |\nabla \ln f|^2)(x_k) \leq 0 \quad \text{and} \quad \lim_{k\to+\infty} |\nabla \ln f|^2(x_k) = \sup_B |\nabla \ln f|^2.$$

Thus, evaluating (2.23) along $\{x_k\}$ and taking $\limsup$ as $k \to +\infty$, we get

$$\lambda \sup_B |\nabla \ln f|^2 = 0,$$

which is a contradiction because $\lambda > 0$ and f is nonconstant.

Assume now that $|\nabla \ln f| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$. Since $\lambda > 0$ and $\mu \geq 0$, from (2.20) we obtain that $|\nabla \ln f|\Delta_\psi |\nabla \ln f| \geq 0$. Applying Corollary 2.1, $|\nabla \ln f|$ must be a constant. Using this latter fact into (2.20) we obtain that f is a constant, which is again a contradiction. $\square$

2.2.2 Proof of the Main Results

Let $M = B \times_f F$ be a gradient Ricci soliton warped product with warping function f. By Lemma 2.1, we can assume without loss of generality that the potential function is the lift $\tilde{\varphi} = \varphi \circ \pi$ of a smooth function φ on B to M. By Proposition 2.1, the base space B is a Ricci-Hessian type manifold satisfying (2.4) and (2.5), while the fiber F is an Einstein manifold with Ricci tensor $\mathrm{Ric}_F = \mu g_F$, where μ is given by (2.6). Now, we are in a position to give the proof of the main results of this chapter. We start by proving a triviality result in the steady case.

Theorem 2.1 *Let $B\times_f F$ be a gradient steady Ricci soliton with fiber having nonnegative scalar curvature. Then, it must be a standard Riemannian product provided the warping function satisfies $f \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

Proof: Since $\lambda = 0$ and $\mu \geq 0$, we use part b) of Proposition 2.6 to conclude that f must be a constant and $\mu = 0$. This completes our proof. □

Theorem 2.2 *It is not possible to construct a gradient expanding Ricci soliton warped product $B^n \times_f F^m$ with fiber having nonnegative scalar curvature and warping function satisfying either of the following conditions: $f \in L^\infty(B)$ or $f \in L^p(B^n, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

Proof: Since $\lambda < 0$ and $\mu \geq 0$, part c) of Proposition 2.6 shows that there is no a Ricci-Hessian type manifold B satisfying the additional Eq. (2.5) with the parameter function f satisfying one of the following conditions: $f \in L^\infty(B)$ or $f \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$. This concludes the proof. □

As an application of Propositions 2.7 and 2.8 we have the following results for gradient Ricci soliton warped products.

Theorem 2.3 *Let $M = B^n \times_f F^m$ be a gradient expanding Ricci soltion with potential function $\tilde{\varphi}$. Then, M is a trivial soliton provided that φ satisfies either of the following conditions: $|\nabla\varphi| \in L^\infty(B)$ or $|\nabla\varphi| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

Proof: Since $\lambda < 0$, we use Proposition 2.7 to conclude that φ is constant if it satisfies either $|\nabla\varphi| \in L^\infty(B)$ or $|\nabla\varphi| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$. Thus, M must be a trivial soliton, that is, an Einstein manifold. □

Theorem 2.4 *It is not possible to construct a gradient shrinking Ricci soliton $B^n \times_f F^m$ with fiber having nonpositive scalar curvature and warping function satisfying either of the following conditions: $|\nabla \ln f| \in L^\infty(B)$ or $|\nabla \ln f| \in L^p(B, e^{-\psi}\mathrm{dvol})$ for some $1 < p < +\infty$.*

Proof: Since $\lambda > 0$, Proposition 2.8 shows that there is no Ricci-Hessian type manifold satisfying the additional Eq. (2.5) with the parameter function f satisfying either of the following conditions : $|\nabla \ln f| \in L^\infty(B)$ or $|\nabla \ln f| \in L^p(B, e^{-\psi}d\mathrm{vol})$ for some $1 < p < +\infty$. This concludes the proof. □

2.3 Concluding Remarks

We also prove, under some additional condition, scalar curvature estimates for Ricci-Hessian type manifolds. They follow by combining Proposition 2.4 with following "a-priori" estimate for weak solutions of semi-linear elliptic inequalities under volume assumptions.

Proposition 2.9 (Pigola et al. [15]) *Let $(B, g_B, e^{-h}d\mathrm{vol})$ be a complete weighted manifold. Let $a(x), b(x) \in C^0(B)$, set $a_-(x) = \max\{-a(x), 0\}$ and assume that*

$$\sup_B a_-(x) < +\infty$$

and

$$b(x) \geq \frac{1}{Q(r(x))} \quad \text{on } B,$$

for some positive, non decreasing function $Q(t)$ such that $Q(t) = o(t^2)$, as $t \to +\infty$. Assume furthermore that, for some $H > 0$,

$$\frac{a_-(x)}{b(x)} \leq H \quad \text{on } B.$$

Let $u \in Lip_{loc}(B)$ be a nonnegative solution of

$$\Delta_h u \geq a(x)u + b(x)u^\sigma,$$

weakly on $(B, e^{-h}d\mathrm{vol})$, with $\sigma > 1$. If

$$\liminf_{r\to+\infty} \frac{Q(r)\ln \mathrm{vol}_h(B_r)}{r^2} < +\infty,$$

then, $u(x) \leq H^{\frac{1}{\sigma-1}}$ on B.

By using Proposition 2.9 we have the validity of the next corollary.

Corollary 2.2 *Let (B, g_B, ψ) be a Ricci-type manifold and let $u \in Lip_{loc}(B)$ be a non-negative weak solution of*

$$\Delta_\psi u \geq au + bu^\sigma,$$

for some constants $a \in \mathbb{R}, b > 0$ and $\sigma > 1$. Then,

$$u(x)^{\sigma-1} \leq \frac{\max\{-a, 0\}}{b} \quad on\ B.$$

As an application of Proposition 2.4, we prove the required result that is similar to known theorems in the setting of gradient Ricci solitons proved by Pigola et al. [15, Theorem 3] as well as of m-quasi-Einstein metrics proved by Rimoldi [11, Theorem 3].

Theorem 2.5 *Let (B^n, g_B, ψ) be a Ricci-Hessian type manifold with scalar curvature S. Let us define $S_* = \inf_B S$ and we assume that*

$$\mathrm{Ric}(\nabla \ln f, \nabla \ln f) \leq -\frac{|\nabla^2 \ln f|^2}{2}. \tag{2.24}$$

a) For $\lambda > 0$, one has $0 \leq S_* \leq n\lambda$.

b) For $\lambda = 0$, we have $S_* = 0$. Moreover, either $S > 0$ or $S \equiv 0$. In the latter case, either both f and φ are constant, or (B^n, g_B) is isometric to the Riemannian product $\mathbb{R} \times \Sigma^{n-1}$, where Σ is a Ricci-flat totally geodesic hypersurface of (B^n, g_B).

c) For $\lambda < 0$, we have $n\lambda \leq S_* \leq 0$, and $S > n\lambda$ unless B is an Einstein manifold.

Proof: We start by showing that $S_* = \inf_B S > -\infty$. By part (4) of Lemma 2.2 and by equation (2.24), we have

$$\frac{1}{2}\Delta_\psi S \leq -\frac{1}{n}S^2 + \lambda S. \tag{2.25}$$

Let $S_-(x) = \max\{-S(x), 0\}$. Then, S_- solves weakly the following

$$\frac{1}{2}\Delta_\psi S_- = -\frac{1}{2}\Delta_\psi S \geq \frac{1}{n}(S_-)^2 + \lambda S_-.$$

We apply Corollary 2.2 to obtain that S_- is bounded from above, or equivalently, $S_* = \inf_B S > -\infty$. By the weak maximum principle at infinity for the ψ-Laplacian there exists a sequence $\{x_k\} \subset B$ such that

$$\lim_{k\to+\infty} S(x_k) = S_* \quad \text{and} \quad \liminf_{k\to+\infty} (\Delta_\psi S)(x_k) \geq 0.$$

Taking lim inf of (2.25) at the points of $\{x_k\}$ shows that

$$\frac{S_*}{n}(\lambda n - S_*) \geq 0. \tag{2.26}$$

Now, we distinguish three cases. For $\lambda > 0$, it is immediate from (2.26), that

$$0 \leq S_* \leq n\lambda.$$

This proves part a). To prove part (b), assume $\lambda = 0$. From (2.26) we conclude that $S_* = 0$. According to (2.25), we note that $\Delta_\psi S \leq 0$. Therefore, by the minimum principle either $S > 0$ on B or $S \equiv 0$. If $S \equiv 0$, substituting it into part (4) of Lemma 2.2 we obtain that B is Ricci-flat, and (2.24) yields $\nabla^2(\ln f) = 0$. So, if f is nontrivial then by the Cheeger-Gromoll's argument B is isometric to $\mathbb{R} \times \Sigma^{n-1}$, where Σ is a Ricci-flat totally geodesic hypersurface of (B^n, g_B). Otherwise, when f is constant, if φ is nontrivial we apply the same argument to conclude that B is isometric to $\mathbb{R} \times \Sigma$.

Next, assume $\lambda < 0$. From (2.26) we deduce that $n\lambda \leq S_* \leq 0$. Suppose $S(x_0) = n\lambda$ for some $x_0 \in B$. Since the nonnegative function $w(x) = S(x) - n\lambda$ satisfies

$$\frac{1}{2}\Delta_\psi w \leq -\frac{1}{n}w^2 - \lambda w \leq -\lambda w,$$

and w attains its minimum $w(x_0) = 0$, it follows from the minimum principle that w vanishes identically. Hence, $S \equiv n\lambda$ is a constant, and substituting this into part (4) of Lemma 2.2, we get that B is an Einstein manifold. □

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