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Mean Curvature Flow in an Extended Ricci Flow Background

Matheus Hudson Gama dos Santos

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by

Matheus Hudson Gama dos Santos

Advisor

Professor José Nazareno Vieira Gomes

Co-advisor

Professor Marcus Antonio Mendonça Marrocos

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Prof. Dr. Ronaldo Freire de Lima - UFRN - Membro Externo

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Av. General Rodrigo Octávio, 6200 - Bairro Coroado 1 Campus Universitário Senador Arthur Virgílio Filho, Setor Norte - Telefone: (92) (92) 3305-1181 / Ramal 2405
CEP 69080-900 Manaus/AM - pos-matematica@ufam.edu.br

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“A guerra é sua”.

(José Nazareno)

Resumo

Consideramos funcionais relacionados ao fluxo da curvatura média em um espaço ambiente que evolui por um fluxo de Ricci estendido, dando continuidade a uma perspectiva introduzida por Lott em seu artigo sobre o fluxo da curvatura média em um espaço ambiente que evolui pelo fluxo de Ricci. Focamos principalmente em uma versão estendida ponderada da ação de Gibbons-Hawking-York sobre métricas Riemannianas em variedades compactas com bordo. Calculamos suas propriedades variacionais, a partir do qual surgem naturalmente as condições de bordo para analisar a derivada tempo sobre um fluxo de Ricci-Perelman estendido modificado. Nesta fórmula de derivada tempo aparece uma extensão da expressão diferencial de Harnack-Hamilton. Obtemos equações de evolução para a segunda forma fundamental e a curvatura média em um fluxo de Ricci estendido. No caso especial de solitons gradientes, discutimos solitons de curvatura média e uma monotonicidade tipo Huisken. Mostramos como construir uma família de solitons de curvatura média e uma caracterização de tal família. Finalmente, apresentamos exemplos de solitons de curvatura média em um fluxo de Ricci estendido.

Palavras-chave: Ação de Gibbons-Hawking-York, Fluxo de Ricci estendido, Fluxo da curvatura média, Monotonicidade tipo Huisken.

Abstract

We consider functionals related to mean curvature flow in an ambient space which evolves by an extended Ricci flow from the perspective introduced by Lott when studying mean curvature flow in a Ricci flow background. Mainly, the functional we focus on the Gibbons-Hawking-York action on Riemannian metrics in compact manifolds with boundary. We compute its variational properties, from which naturally arise boundary conditions to the analysis of its time-derivative under Perelman's modified extended Ricci flow. In this time-derivative formula an extension of Hamilton's differential Harnack expression on the boundary integrand appears. We also derive the evolution equations for both the second fundamental form and the mean curvature under mean curvature flow in an extended Ricci flow background. In the special case of gradient solitons to the extended Ricci flow, we discuss mean curvature solitons and establish Huisken's monotonicity-type formula. We show how to construct a family of mean curvature solitons and establish a characterization of such a family. Finally, we present examples of mean curvature solitons in an extended Ricci flow background.

Keywords: Gibbons-Hawking-York action, Extended Ricci flow, Mean curvature flow, Huisken-type monotonicity.

Contents

Introduction

One of the greatest mathematical achievements of this century was the proof of Thurston's Geometrization Conjecture, by Perelman, which, as a consequence, settled affirmatively the celebrated Poincaré's Conjecture. The main tool used by Perelman in his proofs was the *Ricci flow*, introduced by Hamilton [Ham82], which is defined as follows. Let $\mathcal{G} := \{g(t)\}_{t\in[a,b]}$ be a smoothly varying family of Riemannian metrics on an $(n \geqslant 3)$-dimensional compact smooth manifold M. One says that $\mathcal{G}$ satisfies the Ricci flow equation if

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} \;\; \forall t \in [a,b], \tag{1}$$

where $\mathrm{Ric}_{g(t)}$ denotes the Ricci curvature of $(M, g(t))$.

Hamilton established the existence and the uniqueness of solutions to (1) in a maximal interval $[0,T)$, $T \leqslant +\infty$, for any given initial metric $g = g(0)$. This maximal solution is then called the *Ricci flow* with initial condition g, and T (whenever finite) is called the *blow-up time* of the flow.

The standard example of a Ricci flow is the family $\mathcal{G} := \{g(t)\}_{t\in[0,T)}$ of metrics on the 3-sphere $\mathbb{S}^3$ with $g(0) = r_0^2 g_{\mathbb{S}^3}$ and $g(t) = (r_0^2 - 4t) g_{\mathbb{S}^3}$, where $r_0 > 0$ and $g_{\mathbb{S}^3}$ denotes the standard Euclidean metric of $\mathbb{S}^3$. One then verifies that the blow up time of this flow is $T = r_0^2/4$.

An important geometric flow also considered by Hamilton was the celebrated *mean curvature flow* (MCF, for short), which falls in the class of *extrinsic geometric flows* (see definition below). Since then, mean curvature flow has been a constant object of investigation and has experienced great development in the last decades. It should also be mentioned that MCF has applications in many fields, including geometric analysis, geometric measure theory, and partial differential equations, to name a few.

A significant contribution given by Perelman to the study of the Ricci flow was the discovery of its gradient-like structure, namely, he showed how the Ricci flow can be regarded as a gradient flow from its $\mathcal{F}$-functional on compact manifolds with weighted preserving-measure, for details see [Per02, Sects. 1 and 3] and [KL08, Sects. 10 and 12]. Moreover, he defined an associated entropy by means of its $\mathcal{W}$-functional (see Section 1.2).

In a similar way, List showed how the extended Ricci flow can also be regarded as a gradient flow (cf. [Lis08, Lem. 3.4 and Thm. 6.1]). Moreover, he proved the existence of a Perelman $\mathcal{F}$-type functional such that the stationary points are solutions to the static Einstein vacuum equations and studied an extended parabolic system which is equivalent to the gradient flow of his functional. We will see that, in the boundary case, interesting properties arise when the boundary evolves by some geometric flow.

In [Eck07], Ecker defined a version of Perelman's $\mathcal{W}$-functional for Ricci flow on bounded domains with smooth boundary. Curiously, in its time-derivative formula, it appears Hamilton's differential Harnack expression on the boundary integrand. It should also be mentioned that, in this work, Ecker conjectured that his functional is nondecreasing in time under mean curvature flow of any compact hypersurface in $\mathbb{R}^n$, see (1.25) and Prop. 2.19 for the definition of Ecker's functional and its time-derivative.

Inspired by Ecker's work, Lott [Lot12] approached mean curvature flow in arbitrary Ricci flow background by introducing an analogue of Perelman's $\mathcal{F}$-functional for a manifold M with boundary ∂M. More precisely, he added a boundary term to the interior integral of $\mathcal{F}$, obtaining then a weighted version I_∞ of the Gibbons-Hawking-York action [GH77, Yor72], see also Araújo [Ara03]. In a similar way, one can think of an analogous conjecture for weighted Gibbons-Hawking-York action I_∞ under the mean curvature flow in a Ricci flow background, which is still an open problem. In both cases, an answer for these open problems required a study on the boundary integrand of the time-derivative of these functionals. In this setting, the main results obtained by Lott include the determination of the evolution equations of the action I_∞, of the second fundamental form of ∂M, and of the mean curvature of ∂M under Perelman's modified Ricci flow.

In this thesis, we intend to consider Lott's program in the context of mean curvature flow in an extended Ricci flow background. To be more precise, let M be an $n(\geqslant 3)$-dimensional smooth manifold and let $(g(t),w(t))$ be a solution to the *extended Ricci flow*

$$\begin{cases} \frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}_{g(t)} + 2\alpha_n\,\mathrm{d}w(t)\otimes \mathrm{d}w(t), \\ \frac{\partial}{\partial t}w(t) = \Delta_{g(t)}w(t), \end{cases} \tag{2}$$

in $M\times[0,T)$, for some initial value (g,w). Here and throughout this thesis, $\alpha_n = (n-1)/(n-2)$, $\mathrm{Ric}_{g(t)}$ stands for the Ricci tensor of the Riemannian metric $g(t)$, the Laplacian operator $\Delta_{g(t)}$ is the trace of the Hessian operator $\mathrm{Hess}_{g(t)}$, and $\mathrm{d}w(t)\otimes\mathrm{d}w(t)$ denotes the tensor product of the 1-form $\mathrm{d}w(t)$ by itself, which is metrically dual to gradient vector field $\nabla w(t)$ computed on $g(t)$ of a scalar smooth function $w(t)$ on M. For an account of extended Ricci flows, including proof of short-time existence of solutions to (2) on complete manifolds, we refer to [Lis08, Thm. 4.1]. In this paper, List also showed that Hamilton's Ricci flow and the static Einstein vacuum equations are closely connected by extended Ricci flow, which justifies the value of the constant α_n. So, he provided an interesting and useful link from problems in low-dimensional topology and geometry to physical questions in general relativity.

A gradient soliton to the extended Ricci flow is, by definition, a self-similar solution

$(\overline{g}(t), \overline{w}(t))$ of (2) given by

$$\begin{cases} \overline{g}(t) = \sigma(t)\psi_t^* g, \\ \overline{w}(t) = \psi_t^* w, \end{cases}$$

for some initial value (g, w), where ψ_t is a smooth one-parameter family of diffeomorphisms of M generated from the flow of $\nabla_g f/\sigma(t)$ computed on g, for some $f \in C^\infty(M)$, and σ is a smooth positive function on t. By setting $\overline{f}(t) = \psi_t^* f$, the system (2) becomes

$$\begin{cases} \mathrm{Ric}_{\overline{g}} + \mathrm{Hess}_{\overline{g}} \overline{f} - \alpha_n \, d\overline{w} \otimes d\overline{w} = \frac{c}{2t} \overline{g}, \\ \Delta_{\overline{g}} \overline{w} = \langle \nabla_{\overline{g}} \overline{f}, \nabla_{\overline{g}} \overline{w} \rangle_{\overline{g}}, \end{cases}$$

where $c = 0$ in the steady case (for $t \in \mathbb{R}$ and $\psi_0 = \mathrm{Id}$), $c = -1$ in the shrinking case (for $t \in (-\infty, 0)$ and $\psi_{-1} = \mathrm{Id}$) and $c = -1$ in the expanding case (for $t \in (0, \infty)$ and $\psi_1 = \mathrm{Id}$). Moreover,

$$\frac{\partial}{\partial t} \overline{f} = \|\nabla_{\overline{g}} \overline{f}\|_{\overline{g}}^2.$$

The function $\overline{f}$ is then called the *potential function*. For details and proofs, see Subsection 2.3.3.

We shall consider mean curvature flows in the following context: let $(g(t), w(t))$ be an extended Ricci flow in $M \times [0, T)$. Given an $(n-1)$-dimensional smooth compact manifold Σ without boundary, let $\{x(\cdot, t);\ t \in [0, T)\}$ be a smooth one-parameter family of immersions of Σ into M. For each $t \in [0, T)$, set $x_t = x(\cdot, t)$ and Σ_t for the hypersurface $x_t(\Sigma)$ of $(M, g(t))$, i.e.,

$$\Sigma_t := (x_t(\Sigma), g(t)),\ t \in [0, T),$$

and suppose that the family $\mathscr{F} := \{\Sigma_t\,;\, t \in [0, T)\}$ evolves under mean curvature flow

$$\begin{cases} \frac{\partial}{\partial t} x(p, t) = H(p, t) e(p, t), \\ x(p, 0) = x_0(p), \end{cases}$$

where $H(p, t)$ and $e(p, t)$ are the mean curvature and the unit normal of Σ_t at the point p in Σ, respectively. In this setting, we say that $\mathscr{F}$ is the *mean curvature flow in the* $(g(t), w(t))$-*extended Ricci flow background*. In the particular case $(g(t), w(t)) = (\overline{g}(t), \overline{w}(t))$ is a gradient soliton to the extended Ricci flow on M with potential function $\overline{f}$, a hypersurface $\Sigma_t \in \mathscr{F}$ is a *mean curvature soliton*, if

$$H(p, t) + e(p, t)\overline{f} = 0 \ \forall p \in \Sigma_t.$$

Here, $e(\,\cdot\,, t)$ must be the inward unit normal vector field on Σ_t.

Now suppose that M is an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary

∂M. Let $\mathrm{met}(M)$ be the set of all metrics g on M. We define the *weighted extended Gibbons-Hawking-York (GHY, for short) action* $I_\infty^{\alpha_n}$ on the product $\mathscr{P}(M) := \mathrm{met}(M) \times C^\infty(M) \times C^\infty(M)$ as follows

$$I_\infty^{\alpha_n}(g,f,w) = \int_M \left(R_\infty - \alpha_n|\nabla w|^2\right)e^{-f}\,\mathrm{dV} + 2\int_{\partial M} H_\infty e^{-f}\,\mathrm{dA}, \tag{3}$$

where $R_\infty = R_g + 2\Delta_g f - |\nabla f|^2$ is the *weighted scalar curvature* of g, the function $H_\infty = H + e_0 f$ is the *weighted mean curvature* with respect to the inward unit normal field e_0 on ∂M, the forms dV and dA are the n-dimensional Riemannian measure of (M,g), and the $(n-1)$-dimensional Riemannian measure of $(\partial M, g)$, respectively.

The action $I_\infty^{\alpha_n}$ is the proper extension to our context of the action I_∞ introduced by Lott in [Lot12]. It should also be mentioned that the function R_∞ arises quite naturally, as observed by Perelman [Per02, Sect. 1.3], and H_∞ is in fact the appropriate geometric object when we are using a weighted measure (see, e.g., [Gro03, Sect. 9.4.E] and Section 1.1.4).

Our first main result extends [Lot12, Thm. 1] to the context of mean curvature flow in an extended Ricci flow background. It reads as follows (see Section 1.1 for definitions and notation).

Theorem 1. Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and let $I_\infty^{\alpha_n}$ be the weighted extended GHY-action on $\mathscr{P}(M)$ defined as in (3). Suppose that the family $\{\partial M_t\,;\, t \in [0,T)\}$ is a MCF in the $(g(t),w(t))$-extended Ricci flow background which satisfies $e_0 w = 0$ on ∂M, where e_0 is the inward unit normal vector field on ∂M. Under these conditions, if $u := e^{-f}$ is a solution to the conjugate heat equation

$$\frac{\partial}{\partial t}u = -\Delta_g u + R_g u - \alpha_n |\nabla w|^2 u \tag{4}$$

in $M \times [0,T)$, with $e_0 u = Hu$ on ∂M, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I_\infty^{\alpha_n} = {} & 2\int_M \left(|\mathrm{Ric} + \mathrm{Hess}\, f - \alpha_n\, \mathrm{d}w \otimes \mathrm{d}w|^2 + \alpha_n\left(\Delta_g w - \langle \nabla w, \nabla f\rangle\right)^2\right) e^{-f}\,\mathrm{dV} \\
& + 2\int_{\partial M} \Big(\frac{\partial}{\partial t}H - 2\langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f, \widehat{\nabla} f) + 2R^{0i}\widehat{\nabla}_i f - \frac{1}{2}\nabla_0 R_g - HR_{00} \\
& + \alpha_n \mathcal{A}(\widehat{\nabla} w, \widehat{\nabla} w)\Big) e^{-f}\,\mathrm{dA},
\end{aligned}$$

where $\mathcal{A}$ is the second fundamental form of ∂M and $\widehat{\nabla} f$ denotes the gradient of f on $(\partial M, g(t))$.

For the proof of Theorem 1, we first study the *Perelman's modified extended Ricci flow* (see Subsection 2.2.1), and then "translate" the results for the context of extended Ricci flow. Also, as an application of Theorem 1, we obtain an extension of Hamilton's differential Harnack expression for the mean curvature flow in Euclidean space to the more general context of mean curvature flow in a gradient steady soliton background (cf. Corollary 2.21).

Now let us consider an $n(\geqslant 3)$-dimensional smooth manifold M, and let $(\overline{g}(t), \overline{w}(t))$ be a gradient soliton to the extended Ricci flow on M for some initial value (g, w) and with potential function $\overline{f} = \psi_t^* f$, where $\{\psi_t\}$ is the smooth one-parameter family of diffeomorphisms of M generated by $Y_t = \nabla_g f / \sigma(t)$, with $\sigma(t) = -\kappa t$ and $\psi_{-\kappa} = \mathrm{Id}$, where $\kappa = 1$ in the shrinking case (for $t \in (-\infty, 0)$), $\kappa = -1$ in the expanding case (for $t \in (0, +\infty)$) and $\sigma(t) = 1$ in the steady case (for $t \in \mathbb{R}$) with $\psi_0 = \mathrm{Id}$. Besides, let $\{x(\,\cdot\,, t)\}$ be a smooth one-parameter family of immersions of an $(n-1)$-dimensional smooth compact manifold Σ without boundary into M, given by $x(\cdot, t) := \psi(\cdot, -t - 2\kappa)$ and $x(\cdot, t) := \psi(\cdot, -t)$ in the steady case. For each t, set $\Sigma_t := (x_t(\Sigma), \overline{g}(t))$ for the hypersurface of $(M, \overline{g}(t))$, and $\mathscr{G} := \{\Sigma_t\}$. With this setting in mind, we show how to construct a family of mean curvature solitons and establish a characterization of such a family. This is the content of our second main result.

Theorem 2. If Σ is the f-minimal hypersurface of (M, g), then $\mathscr{G}$ is a family of mean curvature solitons. Besides, any family $\mathscr{F}$ of mean curvature solitons is given by $\mathscr{G}$ up to reparametrization.

Our third main result is Huisken's monotonicity-type formula for the mean curvature flow in an extended Ricci flow background, as stated below.

Theorem 3. Let M be an $n(\geqslant 3)$-dimensional smooth manifold. Let $(\overline{g}(t), \overline{w}(t))$ be a gradient soliton to the extended Ricci flow on M with potential function $\overline{f}$. Assume that $\mathscr{F} := \{\Sigma_t\}$ is a MCF in the $(\overline{g}, \overline{w})$-extended Ricci flow background, denote by $\mathrm{dA}_{\overline{g}}$ the $(n-1)$-dimensional Riemannian measure on Σ_t induced by $\overline{g}(t)$, and set $\mathrm{Area}_{\overline{f}}(\Sigma_t) := \int_{\Sigma_t} e^{-\overline{f}}\, \mathrm{dA}_{\overline{g}}$. Under these conditions, the function $\Phi(t)$ given by:

(i) $\mathbb{R} \ni t \mapsto \mathrm{Area}_{\overline{f}}(\Sigma_t)$ in the steady case,

(ii) $(-\infty, 0) \ni t \mapsto (-t)^{-(n-1)/2}\, \mathrm{Area}_{\overline{f}}(\Sigma_t)$ in the shrinking case, and

(iii) $(0, \infty) \ni t \mapsto t^{-(n-1)/2}\, \mathrm{Area}_{\overline{f}}(\Sigma_t)$ in the expanding case,

is nonincreasing. Moreover, $\Phi(t)$ is constant if and only if $\mathscr{F}$ is a family of mean curvature solitons.

It is worth mentioning here a result by Huisken, from which we know that the shrinking self-similar solutions to the mean curvature flow in Euclidean space are exactly singularity of type-I (i.e., the growth rate estimate for the norm of the second fundamental form is bounded) and asymptotically self-similar which appears as stationary points for the Gaussian area-type functional playing the role of the energy-type functional (see [Hui90] for details). Theorem 3 is a useful tool for studying an analogous to Huisken's result to mean curvature solitons in the $(\overline{g}, \overline{w})$-extended Ricci flow background.

We point out that, by considering particular cases of our results (for instance, assuming $g(t)$ or $w(t)$ constant), we recover some previous results on mean curvature flows (see Remarks 2.7, 2.15, 2.18, 2.22 and 2.27).

The thesis is organized as follows. In Chapter 1, we fix some notation and formulae. We compute the Laplacian of the second fundamental form on the line of [Hui86], see Subsection 1.1.2. We discuss both scalar curvature and mean curvature in the context of weighted smooth manifolds, showing that the former arises from the weighted Schrödinger-Lichnerowicz formula to a weighted Dirac operator (Subsection 1.1.4), whereas the latter is closely related to f-minimal hypersurfaces (Section 1.1.3). We also discuss the evolution of the weighted GHY-action and derive some actions in terms of it (Section 1.2). In Chapter 2, we introduce Perelman's modified extended Ricci flow (Section 2.2), determine its evolution equations (Subsection 2.2.2) and provide the proofs of Theorems 1, 2 and 3 (Section 2.3). In Chapter 3, we consider an $(n \geqslant 3)$-dimensional smooth compact manifold M with boundary ∂M and define a Perelman's Entropy-type functional on $\mathscr{P}(M) \times \mathbb{R}_+$. Next, we prove a version of Theorem 1 for this functional. In particular, by considering M compact without boundary, we recover the result by List [Lis08, Thm. 6.1].

Chapter 1

The weighted Gibbons-Hawking-York action

In this chapter, we take a closer look at the weighted Gibbons-Hawking-York (GHY, for short) action from which we derive Perelman's $\mathcal{F}$ and $\mathcal{W}$ type functionals on smooth manifolds with boundary. We begin by fixing some notation and reminding the reader of some basic facts about Riemannian geometry. After, we compute Laplacian of the second fundamental form in the line of [Hui86], and we motivate both scalar and mean curvatures on weighted smooth manifolds. We give the proof for the evolution of weighted GHY-action and derive some actions in terms of it.

1.1 Preliminaries

In this thesis, the manifolds are assumed to be orientable and connected. Also, in dealing with flows, we shall usually simplify the notation by suppressing the parameter t. Moreover, we are using the Einstein summation convention.

We shall adopt the following notation. Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M and let $\mathrm{met}(M)$ be the set of all Riemannian metrics on M. Let us denote the local coordinates at $p \in M$ by $\{x_\alpha\}_{\alpha=0}^{n-1}$ and the local coordinate basis by $\{\partial_\alpha\}_{\alpha=0}^{n-1}$, the corresponding dual basis is denoted by $\{\mathrm{d}x^\alpha\}_{\alpha=0}^{n-1}$. Near ∂M, we take x_0 to be a local defining function for ∂M. We denote the local coordinates for ∂M by $\{x_i\}_{i=1}^{n-1}$. We choose these coordinates near a point at ∂M so that $\partial_0|_{\partial M} = e_0$ is the inward unit normal field e_0 on ∂M. The Greek letters $\alpha, \beta, \dots$ stand for the indices associated with the coordinates on M, while $i, j, \dots$ for the indices of the coordinates on ∂M.

For a Riemannian metric $g = \langle , \rangle$ on M we denote by ∇ the Levi-Civita connection on TM and by $\widehat{\nabla}$ the Levi-Civita connection on $T\partial M$. By simplicity, we also denote $\nabla_\alpha = \nabla_{\partial_\alpha}$. As usual, let $g^{\alpha\beta}$ denote the $\alpha\beta$th entry of the inverse of $(g_{\alpha\beta})$ in the basis $\{\partial_\alpha\}_{\alpha=0}^{n-1}$. The Riemannian volume element of g on M is denoted by dV and dA stands for the induced Riemannian area element on ∂M.

In what concerns ∂M (or a hypersurface Σ of M), we write $\mathcal{A}_{ij} = \langle e_0, \nabla_{\partial_j}\partial_i\rangle$ for its second fundamental form and $H = g^{kl}\mathcal{A}_{kl}$ for its mean curvature.

In local coordinates, the Riemannian curvature tensor is given by

$$R^{\zeta}{}_{\alpha\beta\gamma}\partial_\zeta = \mathrm{Rm}(\partial_\alpha, \partial_\beta)\partial_\gamma = \nabla_\beta\nabla_\alpha\partial_\gamma - \nabla_\alpha\nabla_\beta\partial_\gamma$$

so that $R^{\zeta}{}_{\alpha\beta\gamma} = g^{\xi\zeta}R_{\alpha\beta\gamma\xi}$, where $R_{\alpha\beta\gamma\xi} = \langle \mathrm{Rm}(\partial_\alpha, \partial_\beta)\partial_\gamma, \partial_\xi\rangle$. The Riemann tensor, in terms of the Christoffel symbols, is given by

$$R^{\zeta}{}_{\alpha\beta\gamma} = \partial_\beta\Gamma^{\zeta}_{\alpha\gamma} - \partial_\alpha\Gamma^{\zeta}_{\beta\gamma} + \Gamma^{\xi}_{\alpha\gamma}\Gamma^{\zeta}_{\beta\xi} - \Gamma^{\xi}_{\beta\gamma}\Gamma^{\zeta}_{\alpha\xi}.$$

Furthermore, the Ricci tensor and the scalar curvature are given by

$$R_{\alpha\beta} := \mathrm{Ric}_g(\partial_\alpha, \partial_\beta) := g^{\gamma\zeta}R_{\alpha\gamma\beta\zeta} \quad \text{and} \quad R_g := g^{\alpha\beta}R_{\alpha\beta},$$

respectively. We claim that for all V be a vector field on M with components $(v^0, \ldots, v^{n-1})$, we have

$$\nabla_\beta v^\zeta = \partial_\beta v^\zeta + v^\gamma\Gamma^{\zeta}_{\beta\gamma}.$$

In fact, note that

$$\nabla_\beta V = \nabla_\beta\left(v^\gamma\partial_\gamma\right) = (\partial_\beta v^\gamma)\partial_\gamma + v^\gamma\nabla_\beta\partial_\gamma = (\partial_\beta v^\zeta + v^\gamma\Gamma^{\zeta}_{\beta\gamma})\partial_\zeta =: \nabla_\beta v^\zeta\partial_\zeta.$$

The second covariant derivative on M gives us

$$\begin{aligned}\nabla_\alpha\nabla_\beta V &= \partial_\alpha(\nabla_\beta v^\zeta)\partial_\zeta + \nabla_\beta v^\zeta(\nabla_{\partial_\alpha}\partial_\zeta) = (\partial_\alpha\partial_\beta v^\zeta + (\partial_\alpha v^\gamma)\Gamma^{\zeta}_{\beta\gamma} + v^\gamma(\partial_\alpha\Gamma^{\zeta}_{\beta\gamma}) + \nabla_\beta v^\gamma\Gamma^{\zeta}_{\alpha\gamma})\partial_\zeta \\ &=: \nabla_\alpha\nabla_\beta v^\zeta\partial_\zeta,\end{aligned}$$

where

$$\nabla_\alpha\nabla_\beta v^\zeta = \partial_\alpha\partial_\beta v^\zeta + (\partial_\alpha v^\gamma)\Gamma^{\zeta}_{\beta\gamma} + v^\gamma(\partial_\alpha\Gamma^{\zeta}_{\beta\gamma}) + \partial_\beta v^\gamma\Gamma^{\zeta}_{\alpha\gamma} + v^\gamma\Gamma^{\xi}_{\beta\gamma}\Gamma^{\zeta}_{\alpha\xi}.$$

As a consequence, we have

$$\nabla_\alpha\nabla_\beta v^\zeta - \nabla_\beta\nabla_\alpha v^\zeta = -R^{\zeta}_{\alpha\beta\gamma}v^\gamma, \tag{1.1}$$

which is known as Symmetry Lemma.

For a smooth function f on M, we write its gradient as $\nabla f = \nabla^\alpha f\partial_\alpha$ so that $\nabla^\alpha f = g^{\alpha\beta}\nabla_\beta f$ and $|\nabla f|^2 = g^{\alpha\beta}\nabla_\alpha f\nabla_\beta f$, where $\nabla_\beta f = \langle\nabla f, \partial_\beta\rangle$.

For a vector field X on M, we write the divergence of X at g as

$$\mathrm{div}_g(X) = g^{\alpha\beta} g(\nabla_{\partial_\alpha} X, \partial_\beta) = g^{\alpha\beta}\nabla_\alpha X_\beta - g^{\alpha\beta} X_\gamma \Gamma^\gamma_{\alpha\beta},$$

where $X_\beta = g(X, \partial_\beta)$. In particular,

$$\Delta_g f = \mathrm{div}_g(\nabla f) = g^{\alpha\beta}\nabla_\alpha\nabla_\beta f = \nabla^\beta\nabla_\beta f,$$

where

$$\nabla_\alpha\nabla_\beta f := \mathrm{Hess}_g f(\partial_\alpha, \partial_\beta) = g(\nabla_{\partial_\alpha}\nabla f, \partial_\beta) = \partial_\alpha\partial_\beta f - (\nabla_{\partial_\alpha}\partial_\beta) f.$$

is the Hessian of f at g.

We denote $\widehat{\nabla}_i\widehat{\nabla}_j f$ and $\widehat{\Delta} f$ the corresponding Hessian and Laplacian of f computed in the induced metric on ∂M.

For a 2-tensor field T on M we consider its associate operator $\widetilde{T}$ given by the equation $T_{\alpha\beta} = \langle \widetilde{T}(\partial_\alpha), \partial_\beta\rangle = \langle T^\zeta{}_\alpha \partial_\zeta, \partial_\beta\rangle$, where $T^\zeta{}_\alpha$ represents the coordinate of the vector $\widetilde{T}(\partial_\alpha)$. Hence $T^\zeta{}_\alpha = g^{\zeta\gamma} T_{\alpha\gamma}$. By simplicity, we will use $T^{\xi\zeta} = g^{\xi\alpha} g^{\zeta\gamma} T_{\alpha\gamma}$.

We can also consider $\omega(X) = T(X, e_0)$ an 1-form on ∂M with X tangent vector field to ∂M. Thus, we take the covariant derivative of the boundary on ∂M to obtain

$$\begin{aligned}
(\widehat{\nabla}_i\omega)_j &:= \partial_i(\omega(\partial_j)) - \omega(\widehat{\nabla}_i\partial_j)\\
&= \partial_i(T(\partial_j, e_0)) - T(\widehat{\nabla}_{\partial_i}\partial_j, e_0)\\
&= (\nabla_i T)_{j0} + T(\nabla_{\partial_i}\partial_j, e_0) + T(\partial_j, \nabla_{\partial_i} e_0) - T(\widehat{\nabla}_{\partial_i}\partial_j, e_0)\\
&= (\nabla_i T)_{j0} + \mathcal{A}_{ij} T_{00} - g^{kl}\mathcal{A}_{il} T_{jk}.
\end{aligned}$$

Define $\widehat{\nabla}_i T_{j0} = (\widehat{\nabla}_i\omega)_j$. Thus

$$(\nabla_i T)_{j0} = \widehat{\nabla}_i T_{j0} - \mathcal{A}_{ij} T_{00} + \mathcal{A}^k{}_i T_{jk}. \tag{1.2}$$

Let us show that the covariant derivative on M always commutes with the trace of an arbitrary 2-tensor field v on M. For simplicity, we write $\nabla_X v_{\alpha\beta} := (\nabla_X v)_{\alpha\beta}$

Lemma 1.1. *Let $v^\alpha{}_\alpha$ be the trace of a 2-tensor field v on M. Then*

$$X(v^\alpha{}_\alpha) = g^{\alpha\beta}\nabla_X v_{\alpha\beta}.$$

Proof. It is enough to prove the equality to $X = \partial_\gamma$. Since $g^{\alpha\theta} g_{\theta\zeta} = \delta^\alpha{}_\zeta$, we get

$$\delta_\gamma g^{\alpha\beta} = -g^{\alpha\xi}\Gamma^\beta_{\gamma\xi} - g^{\xi\beta}\Gamma^\alpha_{\gamma\xi},$$

and then

$$
\begin{aligned}
g^{\alpha\beta}(\nabla_{\partial_\gamma} v)_{\alpha\beta} &= g^{\alpha\beta}\partial_\gamma v_{\alpha\beta} - g^{\alpha\beta}\Gamma^{\eta}_{\gamma\alpha} v_{\eta\beta} - g^{\alpha\beta}\Gamma^{\eta}_{\gamma\beta} v_{\alpha\eta} \\
&= \partial_\gamma(g^{\alpha\beta} v_{\alpha\beta}) - \partial_\gamma g^{\alpha\beta} v_{\alpha\beta} - g^{\xi\beta}\Gamma^{\alpha}_{\gamma\xi} v_{\alpha\beta} - g^{\alpha\xi}\Gamma^{\beta}_{\gamma\xi} v_{\alpha\beta} \\
&= \partial_\gamma(v^{\alpha}{}_{\alpha}).
\end{aligned}
$$

This finishes the proof of the lemma. □

Recall that Lie derivative to 2-tensor field on M is given by

$$
\big(\mathcal{L}_X T\big)_{ij} = X(T_{ij}) - T([X,\partial_i],\partial_j) - T(\partial_i,[X,\partial_j]) = \big(\nabla_X T\big)_{ij} + T(\nabla_{\partial_i} X,\partial_j) + T(\partial_i,\nabla_{\partial_j} X), \quad (1.3)
$$

where $\big(\nabla_X T\big)_{ij} = X(T_{ij}) - T(\nabla_X \partial_i,\partial_j) + T(\partial_i,\nabla_X \partial_j)$. Take traces in (1.3) together with Lemma 1.1 to get

$$
g^{ij}\Big(\mathcal{L}_X T\Big)_{ij} = \nabla_X(g^{ij} T_{ij}) + g^{ij} T(\nabla_{\partial_i} X,\partial_j) + g^{ij} T(\partial_i,\nabla_{\partial_j} X). \quad (1.4)
$$

Now, we recall the divergence theorem.

Theorem 1.2 (Divergence Theorem). *Let $(M,\langle,\rangle)$ be an oriented Riemannian manifold with boundary. For any compactly supported smooth vector field X on M,*

$$
\int_M \operatorname{div} X \,\mathrm{dV} = \int_{\partial M} \langle X, \widetilde{e_0}\rangle \,\mathrm{dA},
$$

where $\widetilde{e_0}$ is the outward unit normal vector field along ∂M.

Proof.

$$
\int_M \operatorname{div} X\,\mathrm{dV} = \int_M \mathrm{d}(X \lrcorner \mathrm{dV}) = \int_{\partial M} X \lrcorner \mathrm{dV} = \int_{\partial M} (X^\top + X^\perp)\lrcorner \mathrm{dV} = \int_{\partial M} \langle X, \widetilde{e_0}\rangle \widetilde{e_0} \lrcorner \mathrm{dV} = \int_{\partial M} \langle X, \widetilde{e_0}\rangle \,\mathrm{dA},
$$

where "$\lrcorner$" is the interior multiplication, $X^\top$ and $X^\perp$ are the tangential and normal components of X, respectively (see [Lee12, Thm. 16.32] for more details). □

In this thesis, e_0 stands for the inward unit normal vector field along ∂M, thus the divergence theorem has the opposite sign, as follows

$$
\int_M \operatorname{div} X\,\mathrm{dV} = -\int_{\partial M} \langle X, e_0\rangle \,\mathrm{dA}\,.
$$

Hence, for all $u, v \in C^\infty(M)$, integration by parts is given by

$$
\int_M v \Delta u \,\mathrm{dV} = -\int_M \big\langle \nabla u, \nabla v \big\rangle \,\mathrm{dV} - \int_{\partial M} v e_0 u \,\mathrm{dA} \quad (1.5)
$$

and then

$$\int_M \left(v\Delta u - u\Delta v\right) \mathrm{dV} = \int_{\partial M} \left(ue_0 v - ve_0 u\right) \mathrm{dA}\,. \tag{1.6}$$

1.1.1 Evolution of curvature

In this section, we establish how one can formally take the time-derivative of the Riemann tensor and the scalar curvature under arbitrary metric variations. Henceforth, we assume by simplicity that $\nabla_\zeta \nabla_\beta T_{\alpha\gamma} := (\nabla_\zeta \nabla_\beta T)_{\alpha\gamma}$ stands for the second covariant derivative for each 2-tensor field T on M. We also adopt this notation for k-tensor field T on M, $k \in \{3,4\}$. Let $g(t)$ be a smooth one-parameter family of Riemannian metrics on M and the variation of coefficient of the metrics will be denoted by $\frac{\partial}{\partial t} g_{\alpha\beta} = v_{\alpha\beta}$. For more details, see [AH11].

We start noting that $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma$. Taking the derivate on both sides, we have

$$\frac{\partial}{\partial t} g^{\alpha\beta} = -g^{\alpha\gamma} g^{\beta\zeta} v_{\gamma\zeta}. \tag{1.7}$$

Lemma 1.3. *The symmetric* 2*-tensor field* Ric *evolves by*

$$\frac{\partial}{\partial t} R_{\alpha\beta} = \frac{1}{2} g^{\gamma\zeta} \left(\nabla_\zeta \nabla_\beta v_{\alpha\gamma} - \nabla_\alpha \nabla_\beta v_{\zeta\gamma} + \nabla_\zeta \nabla_\alpha v_{\beta\gamma} - \nabla_\zeta \nabla_\gamma v_{\alpha\beta}\right). \tag{1.8}$$

Proof. See, for example, [AH11]. □

Proposition 1.4. *The scalar curvature R_g evolves by*

$$\frac{\partial}{\partial t} R_g = \nabla_\alpha \nabla_\beta v^{\alpha\beta} - \nabla^\alpha \nabla_\alpha v - v^{\alpha\beta} R_{\alpha\beta}. \tag{1.9}$$

Proof. By definition of escalar curvature R_g, equations (1.8) and (1.7) we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} R_g &= \frac{\partial}{\partial t}\left(R_{\alpha\beta} g^{\alpha\beta}\right) \\
&= R_{\alpha\beta} \frac{\partial}{\partial t} g^{\alpha\beta} + g^{\alpha\beta} \frac{\partial}{\partial t} R_{\alpha\beta} \\
&= -g^{\alpha\gamma} g^{\beta\zeta} v_{\gamma\zeta} R_{\alpha\beta} + \frac{1}{2} g^{\alpha\beta} g^{\gamma\zeta} \left(\nabla_\zeta \nabla_\beta v_{\alpha\gamma} - \nabla_\alpha \nabla_\beta v_{\zeta\gamma} + \nabla_\zeta \nabla_\alpha v_{\beta\gamma} - \nabla_\zeta \nabla_\gamma v_{\alpha\beta}\right) \\
&= -g^{\alpha\gamma} g^{\beta\zeta} v_{\gamma\zeta} R_{\alpha\beta} + \frac{1}{2} g^{\gamma\zeta} g^{\alpha\beta} \nabla_\zeta \nabla_\beta v_{\alpha\gamma} - \frac{1}{2} g^{\gamma\zeta} g^{\alpha\beta} \nabla_\alpha \nabla_\beta v_{\zeta\gamma} + \frac{1}{2} g^{\gamma\zeta} g^{\alpha\beta} \nabla_\zeta \nabla_\alpha v_{\beta\gamma} \\
&\quad - \frac{1}{2} g^{\alpha\beta} g^{\gamma\zeta} \nabla_\zeta \nabla_\gamma v_{\alpha\beta}) \\
&= -g^{\alpha\gamma} g^{\beta\zeta} v_{\gamma\zeta} R_{\alpha\beta} + \nabla_\alpha \nabla_\beta v^{\alpha\beta} - \nabla^\alpha \nabla_\alpha v \\
&= -v^{\alpha\beta} R_{\alpha\beta} + \nabla_\alpha \nabla_\beta v^{\alpha\beta} - \nabla^\alpha \nabla_\alpha v.
\end{aligned}$$

This finishes the proof of the proposition. □

1.1.2 Laplacian of the second fundamental form

In this section we compute the Laplacian of the second fundamental form of a hypersurface $(\Sigma, \widehat{\nabla})$ in a target Riemannian manifold (M, ∇). In [Sim68], Simons had computed it for minimal immersions, likewise Huisken [Hui86, Lem. 2.1] had proposed the corresponding formula for any hypersurfaces. Here, we will make the proof of this formula as proposed by Lott [Lot12], namely

$$\begin{aligned}\widehat{\nabla}_i\widehat{\nabla}_j H = (\widehat{\Delta}\mathcal{A})_{ij} + \widehat{\nabla}_i R_{j0} + \widehat{\nabla}_j R_{i0} - \nabla_0 R_{ij} + \mathcal{A}^k{}_i R_{0k0j} + \mathcal{A}^k{}_j R_{0k0i} - \mathcal{A}_{ij} R_{00} + 2\mathcal{A}^{kl} R_{kilj} \\ - HR_{0i0j} - H\mathcal{A}^k{}_i \mathcal{A}_{jk} + \mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij} + \nabla_0 R_{0i0j}. \qquad (1.10)\end{aligned}$$

Recall the well-known Gauss, Ricci, and Codazzi-Mainardi equations,

$$\widehat{R}_{jkim} = R_{jkim} + \mathcal{A}_{ji}\mathcal{A}_{km} - \mathcal{A}_{jm}\mathcal{A}_{ki},$$

$$R^{\perp}_{jk00} = (R_{jk00})^{\perp} + g^{rs}\mathcal{A}_{js}\mathcal{A}_{rk} - g^{rs}\mathcal{A}_{ks}\mathcal{A}_{jr} = 0$$

and

$$R_{0ijk} = \widehat{\nabla}_j\mathcal{A}_{ik} - \widehat{\nabla}_k\mathcal{A}_{ij}, \qquad (1.11)$$

respectively. For a proof, see [DT19, Sect. 1.3]. By definition

$$\begin{aligned}(\widehat{\Delta}\mathcal{A})_{ij} &= g^{kl}\widehat{\nabla}_k\widehat{\nabla}_l\mathcal{A}_{ij} \\ &= g^{kl}\widehat{\nabla}_k(-R_{0ijl} + (\widehat{\nabla}_j\mathcal{A})_{il}) \\ &= -g^{kl}\widehat{\nabla}_k R_{0ijl} + g^{kl}\widehat{\nabla}_k\widehat{\nabla}_j\mathcal{A}_{il}.\end{aligned}$$

Using $(\overline{R}_{jk}\mathcal{A})_{il} = g^{kl}\widehat{\nabla}_k\widehat{\nabla}_j\mathcal{A}_{il} - \widehat{\nabla}_j\widehat{\nabla}_k\mathcal{A}_{il}$ to obtain

$$(\widehat{\Delta}\mathcal{A})_{ij} = -g^{kl}\widehat{\nabla}_k R_{0ijl} + (\overline{R}_{jk}\mathcal{A})_{il} + \widehat{\nabla}_j\widehat{\nabla}_k\mathcal{A}_{il},$$

where $(\overline{R}_{jk}\mathcal{A})_{il} = R^{\perp}_{jk00}\mathcal{A}_{il} - \mathcal{A}(\widehat{R}_{jk}\partial_i, \partial_l) - \mathcal{A}(\partial_i, \widehat{R}_{jk}\partial_l)$, where the Riemann tensor of boundary is given by $\widehat{R}_{jk}\partial_i = \widehat{\nabla}_k\widehat{\nabla}_j\partial_i - \widehat{\nabla}_j\widehat{\nabla}_k\partial_i$ with $\widehat{R}_{jk}\partial_i = (\widehat{R}^l_{jk}\partial_i)\partial_l$, $\widehat{R}^l_{jk}\partial_i = g^{ml}\widehat{R}_{jkim}$. Thus

$$(\widehat{\Delta}\mathcal{A})_{ij} = -g^{kl}\widehat{\nabla}_k R_{0ijl} + g^{kl}(\overline{R}_{jk}\mathcal{A})_{il} + g^{kl}\widehat{\nabla}_j(-R_{0lik} + (\widehat{\nabla}_i\mathcal{A})_{kl}).$$

Now we will compute some terms of this equation. The first one of them is

$$\begin{aligned}(\overline{R}_{jk}\mathcal{A})_{il} &= -g^{ms}(R_{jkim} + \mathcal{A}_{ji}\mathcal{A}_{km} - \mathcal{A}_{jm}\mathcal{A}_{ki})\mathcal{A}_{sl} - g^{ms}(R_{jklm} + \mathcal{A}_{jl}\mathcal{A}_{km} - \mathcal{A}_{jm}\mathcal{A}_{kl})\mathcal{A}_{is} \\ &= -R_{jkim}\mathcal{A}^m{}_l - \mathcal{A}_{ji}\mathcal{A}_{km}\mathcal{A}^m{}_l + \mathcal{A}_{jm}\mathcal{A}_{ki}\mathcal{A}^m{}_l - g^{ms}R_{jklm}\mathcal{A}_{is} - \mathcal{A}_{jl}\mathcal{A}^s{}_k\mathcal{A}_{is} + \mathcal{A}^s{}_j\mathcal{A}_{kl}\mathcal{A}_{is}\end{aligned}$$

where by Guass equation we get $\widehat{R}^s_{jk}\partial_i = g^{ms}\widehat{R}_{jkim} = g^{ms}(R_{jkim} + \mathcal{A}_{ji}\mathcal{A}_{km} - \mathcal{A}_{jm}\mathcal{A}_{ki})$.

Now we are going to proceed as in (1.2). Let T be a 4-tensor field on M, we also can consider $\omega(X,Y,Z) = T(e_0,X,Y,Z)$ a 3-tensor field on ∂M. Again taking the covariant derivative on the boundary, we obtain

$$\begin{aligned}
\widehat{\nabla}_i\omega_{jkl} &:= \partial_i(\omega(\partial_j,\partial_k,\partial_l)) - \omega(\widehat{\nabla}_i\partial_j,\partial_k,\partial_l) - \omega(\partial_j,\widehat{\nabla}_i\partial_k,\partial_l) - \omega(\partial_j,\partial_k,\widehat{\nabla}_i\partial_l) \\
&= \partial_i(T(e_0,\partial_j,\partial_k,\partial_l)) - T(e_0,\widehat{\nabla}_i\partial_j,\partial_k,\partial_l) - T(e_0,\partial_j,\widehat{\nabla}_i\partial_k,\partial_l) - T(e_0,\partial_j,\partial_k,\widehat{\nabla}_i\partial_l) \\
&= (\nabla_i T)_{0jkl} + T(\nabla_i e_0,\partial_j,\partial_k,\partial_l) + T(e_0,\nabla_i\partial_j,\partial_k,\partial_l) + T(e_0,\partial_j,\nabla_i\partial_k,\partial_l) \\
&\quad + T(e_0,\partial_j,\partial_k,\nabla_i\partial_l) - T(e_0,\widehat{\nabla}_i\partial_j,\partial_k,\partial_l) - T(e_0,\partial_j,\widehat{\nabla}_i\partial_k,\partial_l) - T(e_0,\partial_j,\partial_k,\widehat{\nabla}_i\partial_l) \\
&= (\nabla_i T)_{0jkl} + \mathcal{A}_{ij}T_{00kl} + \mathcal{A}_{ik}T_{0j0l} + \mathcal{A}_{il}T_{0jk0} - g^{ms}\mathcal{A}_{is}T_{mjkl}.
\end{aligned}$$

Define $\widehat{\nabla}_i T_{0jkl} = \widehat{\nabla}_i\omega_{jkl}$.

$$\widehat{\nabla}_i T_{0jkl} = (\nabla_i T)_{0jkl} + \mathcal{A}_{ij}T_{00kl} + \mathcal{A}_{ik}T_{0j0l} + \mathcal{A}_{il}T_{0jk0} - g^{ms}\mathcal{A}_{is}T_{mjkl}.$$

The second one of them is

$$\widehat{\nabla}_k R_{0ijl} = (\nabla_k R)_{0ijl} + \mathcal{A}_{ki}R_{00jl} + \mathcal{A}_{kj}R_{0i0l} + \mathcal{A}_{kl}R_{0ij0} - g^{ms}\mathcal{A}_{ks}R_{mijl}.$$

By combining all of the terms, we find that

$$\begin{aligned}
(\widehat{\Delta}\mathcal{A})_{ij} =& -g^{kl}\nabla_k R_{0ijl} - g^{kl}\mathcal{A}_{kj}R_{0i0l} - g^{kl}\mathcal{A}_{kl}R_{0ij0} + g^{kl}g^{ms}\mathcal{A}_{ks}R_{mijl} + g^{kl}\Big(-R_{jkim}\mathcal{A}^m{}_l - \mathcal{A}_{ji}\mathcal{A}_{km}\mathcal{A}^m{}_l \\
&+ \mathcal{A}_{jm}\mathcal{A}_{ki}\mathcal{A}^m{}_l - g^{ms}R_{jklm}\mathcal{A}_{is} - \mathcal{A}_{jl}\mathcal{A}^s{}_k\mathcal{A}_{is} + \mathcal{A}^s{}_j\mathcal{A}_{kl}\mathcal{A}_{is}\Big) - g^{kl}\widehat{\nabla}_j R_{0lik} + g^{kl}(\widehat{\nabla}_j\widehat{\nabla}_i\mathcal{A})_{lk}.
\end{aligned}$$

Since the trace commutes with the covariant derivative on the boundary,

$$g^{kl}(\widehat{\nabla}_j\widehat{\nabla}_i\mathcal{A})_{lk} = \widehat{\nabla}_j\widehat{\nabla}_i(g^{kl}\mathcal{A}_{lk}),$$

and using the contracted Bianchi identity

$$-g^{kl}\nabla_k R_{0ijl} = g^{kl}\nabla_0 R_{ikjl} + g^{kl}\nabla_i R_{k0jl},$$

one has

$$\begin{aligned}
(\widehat{\Delta}\mathcal{A})_{ij} = g^{kl}\nabla_0 R_{ikjl} &+ g^{kl}\nabla_i R_{k0jl} - \mathcal{A}^l{}_j R_{0i0l} - HR_{0ij0} + \mathcal{A}^{lm}R_{mijl} - R_{jkim}\mathcal{A}^{mk} - \mathcal{A}_{ji}\mathcal{A}_{km}\mathcal{A}^{mk} \\
&+ \mathcal{A}_{jm}\mathcal{A}_{ki}\mathcal{A}^{mk} - g^{kl}R_{jklm}\mathcal{A}^m{}_i - \mathcal{A}_{jl}\mathcal{A}^{sl}\mathcal{A}_{is} + H\mathcal{A}^s{}_j\mathcal{A}_{is} - g^{kl}\widehat{\nabla}_j R_{0kli} + \widehat{\nabla}_j\widehat{\nabla}_i H.
\end{aligned}$$

Note that

$$g^{kl}\nabla_0 R_{ikjl} = \nabla_0 R_{ij} - \nabla_0 R_{i0j0} \quad \text{and} \quad g^{kl}\nabla_i R_{k0jl} = -g^{kl}\nabla_i R_{0kjl} = -\nabla_i R_{0j} + \nabla_i R_{00j0},$$

together with

$$-g^{kl}\widehat{\nabla}_j R_{0lik} = -\widehat{\nabla}_j R_{0i} + \nabla_j R_{00i0} \quad \text{and} \quad -g^{kl}R_{jklm} = g^{kl}R_{jkml} = R_{jm} - R_{j0m0}$$

imply

$$\begin{aligned}(\widehat{\Delta}\mathcal{A})_{ij} =& \nabla_0 R_{ij} - \nabla_0 R_{i0j0} - \nabla_i R_{0j} - g^{kl}\mathcal{A}_{kj}R_{0i0l} - g^{kl}\mathcal{A}_{kl}R_{0ij0} + \mathcal{A}^{lm}R_{mijl} - R_{jkim}\mathcal{A}^{mk} - \mathcal{A}_{ji}\mathcal{A}_{km}\mathcal{A}^{mk}\\ &+ \mathcal{A}_{jm}\mathcal{A}_{ki}\mathcal{A}^{mk} + R_{jm}\mathcal{A}^m{}_i - \mathcal{A}^m{}_i R_{j0m0} - \mathcal{A}_{jl}\mathcal{A}^{sl}\mathcal{A}_{is} + H\mathcal{A}^s{}_j\mathcal{A}_{is} - \widehat{\nabla}_j R_{0i} + \widehat{\nabla}_i\widehat{\nabla}_j H.\end{aligned}$$

Hence

$$\begin{aligned}(\widehat{\Delta}\mathcal{A})_{ij} =& \nabla_0 R_{ij} - \nabla_0 R_{i0j0} - \nabla_i R_{0j} - \mathcal{A}^l{}_j R_{0i0l} - HR_{0ij0} + 2\mathcal{A}^{lm}R_{mijl} - \mathcal{A}_{ji}\mathcal{A}_{km}\mathcal{A}^{mk} + \mathcal{A}_{jm}\mathcal{A}_{ki}\mathcal{A}^{mk}\\ &+ R_{jm}\mathcal{A}^m{}_i - \mathcal{A}^m{}_i R_{j0m0} - \mathcal{A}_{jl}\mathcal{A}^{sl}\mathcal{A}_{is} + H\mathcal{A}^s{}_j\mathcal{A}_{is} - \widehat{\nabla}_j R_{0i} + \widehat{\nabla}_j\widehat{\nabla}_i H.\end{aligned}$$

Thus

$$\begin{aligned}(\widehat{\Delta}\mathcal{A})_{ij} =& \nabla_0 R_{ij} - \nabla_0 R_{i0j0} - \widehat{\nabla}_i R_{0j} + \mathcal{A}_{ij}R_{00} - g^{kl}\mathcal{A}_{ik}R_{lj} - \mathcal{A}^l{}_j R_{0i0l} - HR_{0ij0} + 2\mathcal{A}^{lm}R_{mijl}\\ &- \mathcal{A}_{ji}\mathcal{A}_{km}\mathcal{A}^{mk} + R_{jm}\mathcal{A}^m{}_i - \mathcal{A}^m{}_i R_{j0m0} + H\mathcal{A}^s{}_j\mathcal{A}_{is} - \widehat{\nabla}_j R_{0i} + \widehat{\nabla}_j\widehat{\nabla}_i H.\end{aligned}$$

By (1.2) applied to $T = \mathrm{Ric}$ we have

$$\nabla_j R_{0i} = \widehat{\nabla}_j R_{0i} - \mathcal{A}_{ij}R_{00} + g^{kl}\mathcal{A}_{jk}R_{li}$$

and we finally arrive at

$$\begin{aligned}(\widehat{\Delta}\mathcal{A})_{ij} =& \nabla_0 R_{ij} - \nabla_0 R_{i0j0} - \widehat{\nabla}_i R_{0j} + \mathcal{A}_{ij}R_{00} - \mathcal{A}^l{}_j R_{0l0i} + HR_{0i0j} - 2\mathcal{A}^{lm}R_{milj}\\ &- \mathcal{A}_{ji}\mathcal{A}_{km}\mathcal{A}^{mk} - \mathcal{A}^m{}_i R_{0m0j} + H\mathcal{A}^s{}_j\mathcal{A}_{is} - \widehat{\nabla}_j R_{0i} + \widehat{\nabla}_j\widehat{\nabla}_i H,\end{aligned}$$

which is exactly (1.10). Besides, to use forward, take traces in (1.11) to get

$$R_{0j} = \widehat{\nabla}_j H - \widehat{\nabla}_i\mathcal{A}^i{}_j. \tag{1.12}$$

In what follows, we will use the next lemma to describe the Laplacian of a smooth function on the target manifold in terms of isometric immersion objects.

Lemma 1.5. *Let Σ be an n-dimensional Riemannian manifold with Levi-Civita connection $\widehat{\nabla}$ and let (M,g) be an $(m+n)$-dimensional Riemannian manifold with Levi-Civita connection ∇. Given an isometric immersion $f:\Sigma\to M$, then for all $w\in C^\infty(M)$ we have*

$$\Delta_g w = \widehat{\Delta}w - g(\nabla w, \vec{H}) + g^{ab}\nabla_a\nabla_b w,$$

where $\vec{H}$ is the mean curvature vector field on Σ and $a,b\in\{n+1,\dots,n+m\}$.

Proof. Consider a coordinate system $\{\partial_\zeta\}_{\zeta=1}^{n+m}$ so that $\{\partial_i\}_{i=1}^{n}$ is tangent to Σ and $\{\partial_a\}_{a=n+1}^{n+m}$ is normal to Σ, and set $\alpha_{ij} = (\nabla_{\partial_i}\partial_j)^\perp$. Thus for all $w \in C^\infty(M)$ we have

$$\begin{aligned}
\Delta_g w &= g^{\alpha\beta}\Big(\partial_\alpha\partial_\beta w - \big(\nabla_{\partial_\alpha}\partial_\beta\big)w\Big) \\
&= g^{ij}\Big(\partial_i\partial_j w - \big(\nabla_{\partial_i}\partial_j\big)w\Big) + g^{ab}\Big(\partial_a\partial_b w - \big(\nabla_{\partial_a}\partial_b\big)w\Big) \\
&= g^{ij}\Big(\partial_i\partial_j w - \big(\widehat{\nabla}_{\partial_i}\partial_j + \alpha_{ij}\big)w\Big) + g^{ab}\nabla_a\nabla_b w \\
&= \widehat{\Delta} w - g(\nabla w, \vec{H}) + g^{ab}\nabla_a\nabla_b w,
\end{aligned}$$

where $\vec{H} = g^{ij}\alpha_{ij}$ is the mean curvature vector field on Σ. $\square$

1.1.3 Mean curvature flow and f-minimal hypersurfaces

In this subsection, we shall state the classic evolution equations for mean curvature flow in Euclidean spaces. Given an $(n-1)$-dimensional smooth compact manifold Σ without boundary, let $\{x(\cdot,t); t \in [0,T)\}$ be a smooth one-parameter family of immersions of Σ in $\mathbb{R}^n$. Set $\Sigma_t := x_t(\Sigma)$, where $x_t = x(\cdot,t)$, and suppose that the family $\mathscr{F} := \{\Sigma_t\}$ evolves under mean curvature flow, i.e.,

$$\begin{cases} \frac{\partial}{\partial t}x(p,t) = H(p,t)e(p,t), \\ x(p,0) = x_0(p), \end{cases} \tag{1.13}$$

where $H(p,t)$ and $e(p,t)$ are the mean curvature and the unit normal of Σ_t at the point $p \in \Sigma$, respectively. For short-time existence of a solution for (1.13) (see, e.g., the book by Mantegazza [Man11, p. 18]).

If there exist $C > 0$ so that $\sup_{p\in\Sigma}|\mathcal{A}(p,t)| \leqslant C/\sqrt{2(T_{max}-t)}$, we say that MCF is developing at time T_{max} a *type I singularity*. Huisken used a functional (see Remark 2.27) in order to classify these singularities.

Example 1.6. Let $\mathbb{S}^n(R)$ be the n-sphere of radius R, and let $x(p,t) = r(t)x_0(p)$ be a family of immersions of $\mathbb{S}^n(R)$ into $\mathbb{R}^{n+1}$, where $r(t) = \sqrt{R^2 - 2nt} = \sqrt{2n(T_{max}-t)}$ and x_0 is the standard inclusion map. Note that at time $T_{max} = \frac{R^2}{2n}$ the sphere shrinks to a point, so the flow becomes singular (see figure below). Moreover, the norm of the second fundamental form evolves as $|\mathcal{A}(\,\cdot\,,t)| = \frac{\sqrt{n}}{r(t)} = \frac{1}{\sqrt{2(T_{max}-t)}}$.

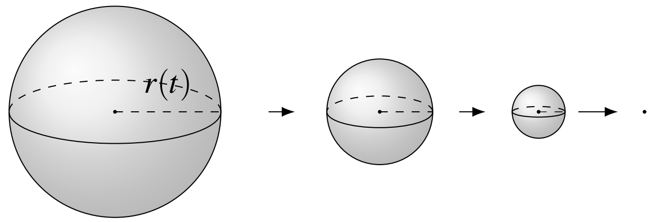

Figure 1.1: Sphere collapsed in finite time.

Another example is given by the cylinders $\mathbb{S}^n(R)\times\mathbb{R}$.

Example 1.7. Let $\mathbb{S}^n(R)\times\mathbb{R}$ be the cylinders, and let $\widetilde{x}(p,s,t)=(x(p,t),s)=(r(t)x_0(p),s)$ be the family of immersions of $\mathbb{S}^n(R)\times\mathbb{R}$ into $\mathbb{R}^{n+2}$, where $r(t)=\sqrt{R^2-2nt}=\sqrt{2n(T_{max}-t)}$ and collapse to $\mathbb{R}$ at time $T_{max}=\frac{R^2}{2n}$. Moreover, $|\mathcal{A}(\cdot,t)|=\frac{\sqrt{n}}{r(t)}=\frac{1}{\sqrt{2(T_{max}-t)}}$.

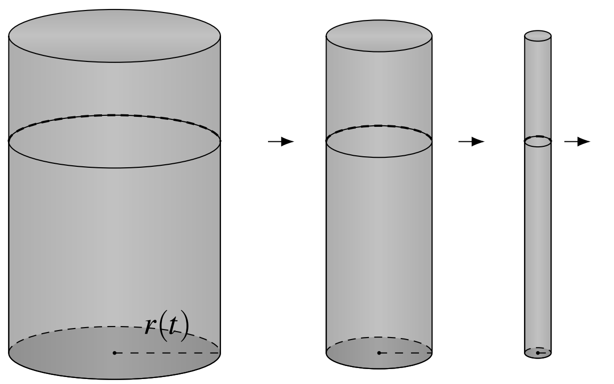

Figure 1.2: As in the Sphere, the Cylinder also collapsed in finite time.

Spheres and cylinders are special examples of *homothetically shrinking flows*, that is, hypersurfaces that simply move by contraction during the evolution by mean curvature.

Proposition 1.8. *Let $\mathscr{F}:=\{\Sigma_t\}$ be a family moving by mean curvature flow in Euclidean space. Then the following evolution equations hold on Σ_t*

$$\frac{\partial}{\partial t}g_{ij}=-2H\mathcal{A}_{ij}, \tag{1.14}$$
$$\frac{\partial}{\partial t}\mathcal{A}_{ij}=(\widehat{\Delta}\mathcal{A})_{ij}-2H\mathcal{A}_{ik}\mathcal{A}^k{}_j+\mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij}, \tag{1.15}$$
$$\frac{\partial}{\partial t}H=\widehat{\Delta}H+\mathcal{A}^{ij}\mathcal{A}_{ij}H, \tag{1.16}$$
$$\frac{\partial}{\partial t}\,\mathrm{dA}=-H^2\,\mathrm{dA}\,. \tag{1.17}$$

For a proof, see [Hui84, Lem. 3.2, Thm. 3.4 and Cor. 3.5] or [Man11, Sect. 2.3].

It is known that minimal submanifolds arise as critical points of the area functional. Recall that a submanifold of the area functional is called *critical* if $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathrm{Area}(X)=0$ for all variational vector field $X = \frac{\partial}{\partial t}\Big|_{t=0} x_t$, where $\{x_t = x(\,\cdot\,,t)\}$ is a smooth one-parameter family of immersions of Σ into M.

For the sake of completeness, we will show an analogous property to f-minimal hypersurfaces which are closely related to this thesis. More precisely, given an n-dimensional Riemannian manifold $(M, g = \langle , \rangle)$, an isometric immersion x of an $(n-1)$-dimensional compact smooth manifold Σ without boundary into M, and a smooth function f on M.

Let $\{x_t = x(\,\cdot\,,t)\}$ be a smooth one-parameter family of immersions of Σ into M, and let $X = \frac{\partial}{\partial t}\Big|_{t=0} x_t$ be the variational vector field along Σ. Let us consider the *f-weighted area functional* given by

$$\mathrm{Area}_f(t) = \int_\Sigma e^{-f\circ x_t}\, \mathrm{dA}_t,$$

where dA_t stands for the area element on $(\Sigma, x_t^* g)$. Recall that $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{dA}_t = \frac{h}{2}\,\mathrm{dA}$, where $h = \mathrm{tr}_g\,\mathcal{H}$ and $\mathcal{H} = \frac{\mathrm{d}}{\mathrm{d}t} x_t^* g$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Area}_f(t) = \int_\Sigma \Big(- \Big\langle \frac{\partial}{\partial t}\Big|_{t=0} x_t, \nabla f \Big\rangle + \frac{h}{2} \Big) e^{-f}\, \mathrm{dA} = \int_\Sigma \Big(- \Big\langle X, \nabla f \Big\rangle + \frac{h}{2} \Big) e^{-f}\, \mathrm{dA}\,.$$

Note that we can write $\mathrm{div}_\Sigma(e^{-f} X^\top) = e^{-f}\,\mathrm{div}_\Sigma(X^\top) - e^{-f}\big\langle \nabla f, X^\top \big\rangle$, so that by divergence theorem and the known identity $\frac{h}{2} = \mathrm{div}_\Sigma(X^\top) - \big\langle X, \vec{H} \big\rangle$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Area}_f(t) = \int_\Sigma \big(- \big\langle \nabla f, X^\perp \big\rangle - \big\langle X, \vec{H} \big\rangle \big) e^{-f}\, \mathrm{dA} = - \int_\Sigma H_\infty \big\langle X, e_0 \big\rangle e^{-f}\, \mathrm{dA},$$

where one sees that the analog of the mean curvature $H_\infty := H + e_0 f$. So the critical points of the f-weighted area functional on Σ are *f-minimal hypersurfaces*, i.e., $H_\infty = 0$. Some results concerning f-minimal hypersurfaces can be found for example in [ALR20], [CZ15], [CVZ21] and [Wei17].

1.1.4 Weighted scalar curvature

Before defining Perelman's functional on a compact smooth manifold M without boundary, let us motivate what R_∞ means for an approach of the weighted manifolds. Given $f \in C^\infty(M)$, consider the smooth metric-measure space $\mathcal{M} = (M, g, e^{-f}\,\mathrm{dV})$. As is now well understood, the analog of the Ricci curvature for $\mathcal{M}$ is the *Bakry-Emery Ricci tensor* $\mathrm{Ric}_\infty = \mathrm{Ric}_g + \mathrm{Hess}_g f$. In [Per02, Sect. 1.3], Perelman pointed out the analog of the scalar curvature, namely

$$R_\infty := R_g + 2\Delta_g f - |\nabla f|^2,$$

which, in fact, works as a scalar curvature on $\mathcal{M}$, since by divergence theorem and the identity $\Delta_g e^{-f} = (|\nabla f|^2 - \Delta_g f)e^{-f}$, one can deduce

$$\int_M (R_g + \Delta_g f)e^{-f}\,\mathrm{dV} = \int_M (R_g + 2\Delta_g f - |\nabla f|^2)e^{-f}\,\mathrm{dV} = \int_M R_\infty e^{-f}\,\mathrm{dV}\,.$$

In order to give a natural interpretation for R_∞, Perelman defined the following *weighted Dirac operator* D_∞ on the spinors $\Sigma^g M$

$$\mathrm{D}_\infty \varphi = \mathrm{D}_g \varphi - \frac{1}{2}\nabla f \cdot \varphi,$$

where D_g denotes the Dirac operator which acts on the spinors bundle $\Sigma^g M$ and the product "$\cdot$" is known as Clifford multiplication (see, e.g., [LM16, Chapt. 2]). There is an equation that relates the Dirac operator D_g with its Ricci tensor, called by $(\frac{1}{2}\,\mathrm{Ric}_g)$-formula (see [KF00, Lem. 1.2]), namely

$$\frac{1}{2}\widetilde{\mathrm{Ric}}_g(X)\cdot\varphi = \mathrm{D}_g(\nabla_X \varphi) - \nabla_X(\mathrm{D}_g \varphi) - \sum_\alpha e_\alpha \cdot \nabla_{\nabla_{e_\alpha} X}\varphi,$$

where $\{e_\alpha\}_\alpha$ is an orthonormal frame on TM. Now the *Bakry-Emery tensor* on $\mathcal{M}$ arises quite naturally when we use to D_∞. In fact,

$$\mathrm{D}_\infty(\nabla_X \varphi) - \nabla_X(\mathrm{D}_\infty \varphi) - \sum_\alpha e_\alpha \cdot \nabla_{\nabla_{e_\alpha} X}\varphi = \frac{1}{2}\big(\widetilde{\mathrm{Ric}}_g(X) + \widetilde{\mathrm{Hess}}_g f(X)\big)\cdot\varphi = \frac{1}{2}\widetilde{\mathrm{Ric}}_\infty(X)\cdot\varphi.$$

The Schrödinger-Lichnerowicz formula which is weaker than $(\frac{1}{2}\,\mathrm{Ric}_g)$-formula in the sense to be a contraction (cf. [Hij01, Lem. 4.11]), is

$$\mathrm{D}_g^2 \varphi = -\nabla^*\nabla\varphi + \frac{1}{4}R_g \cdot \varphi, \tag{1.18}$$

where $\nabla^*\nabla\varphi = -\sum_\alpha \nabla_\alpha \nabla_\alpha \varphi$. Analogously, we have

$$\mathrm{D}_\infty^2 \varphi = \mathrm{D}_g(\mathrm{D}_\infty \varphi) - \frac{1}{2}\nabla f \cdot \mathrm{D}_\infty \varphi = \mathrm{D}_g^2 \varphi - \frac{1}{2}\mathrm{D}_g(\nabla f \cdot \varphi) - \frac{1}{2}\nabla f \cdot \mathrm{D}_g \varphi + \frac{1}{4}\nabla f \cdot \nabla f \cdot \varphi.$$

Since $X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi = -2\langle X, Y\rangle_g \varphi$, and

$$\mathrm{D}_g(\nabla f \cdot \varphi) = \sum_\alpha e_\alpha \cdot \nabla_{e_\alpha}\nabla f \cdot \varphi - \nabla f \cdot \mathrm{D}_g \varphi - 2\nabla_{\nabla f}\varphi,$$

we obtain (see [Hij01, Sect. 4.3])

$$\mathrm{D}_\infty^2 \varphi = \mathrm{D}_g^2 \varphi - \frac{1}{2}\sum_\alpha e_\alpha \cdot \nabla_{e_\alpha}\nabla f \cdot \varphi + \frac{1}{2}\nabla f \cdot \mathrm{D}_g \varphi + \nabla_{\nabla f}\varphi - \frac{1}{2}\nabla f \cdot \mathrm{D}_g \varphi - \frac{1}{4}|\nabla f|^2 \cdot \varphi.$$

Moreover, notice that $\Delta_g f \cdot \varphi = -\sum_\alpha e_\alpha \cdot \widetilde{\mathrm{Hess}}_g(e_\alpha)\cdot\varphi$ (see, e.g., [KF00, p. 132]), and since $R_g \cdot \varphi = -\sum_\alpha e_\alpha \cdot \widetilde{\mathrm{Ric}}_g(e_\alpha)\cdot\varphi$ and (1.18) we can find that R_∞ arises quite naturally of the *weighted Schrödinger-Lichnerowicz* formula

$$\mathrm{D}^2_\infty \varphi = -\nabla^*\nabla\varphi + \nabla_{\nabla f}\varphi + \frac{1}{4}\big(R_g + 2\Delta_g f - |\nabla f|^2\big)\cdot\varphi = -\nabla^*_f\nabla\varphi + \frac{1}{4}R_\infty\cdot\varphi,$$

where $\nabla^*_f\nabla\varphi = \nabla^*\nabla\varphi - \nabla_{\nabla f}\varphi$ can be thought as Drift Laplacian on $\mathcal{M}$. This finishes our remark.

1.2 Perelman's functionals

Let M be a compact smooth manifold without boundary. In [Per02, Rem. 1.3], Perelman defined the $\mathcal{F}$*-functional* on $\mathrm{met}(M)\times C^\infty(M)$ as

$$\mathcal{F}(g,f) = \int_M R_\infty e^{-f}\,\mathrm{dV} \tag{1.19}$$

or, equivalently,

$$\mathcal{F}(g,f) = \int_M (R_g + |\nabla f|^2)e^{-f}\,\mathrm{dV}\,. \tag{1.20}$$

The functional in (1.19) appeared for the first time in the great work of Perelman in the study of the Ricci flow on smooth manifolds without boundary. He had discovered that Ricci flow has a gradient-like structure, i.e., he showed how the Ricci flow can be regarded as a gradient flow on compact manifolds with weighted preserving-measure (see [Per02, Sects. 1 and 3]).

Of course, when M has a boundary, expression (1.20) contains a boundary term. If we add a suitable boundary term for (1.19) then we obtain an extension of $\mathcal{F}$ for the boundary case, which has nicer variational properties (see Eq. (1.27) and Subsection 1.2.2). However, without this additional term, we find in the literature the following result by Cortissoz and Murcia (see [CM19, Prop. 3.1]).

Proposition 1.9 ([CM19, Prop. 3.1])**.** *Let M be an n-dimensional compact smooth manifold with boundary ∂M and $\mathcal{F}:\mathrm{met}(M)\times C^\infty(M)\to\mathbb{R}$ the functional defined as*

$$\mathcal{F}(g,f) = \int_M (R_g + |\nabla f|^2)e^{-f}\,\mathrm{dV}\,.$$

Then its evolution is given by

$$\begin{aligned}\delta\mathcal{F}(v,h) = &\int_M \Big[-v^{\alpha\beta}(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f) + \Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)(R_g - |\nabla f|^2 + 2\Delta_g f)\Big]e^{-f}\,\mathrm{dV}\\ &+ \int_{\partial M}\Big[2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)e_0 f - \Big(\widehat{\nabla}_i v^{i0} - Hv^0{}_0 + \mathcal{A}^{ij}v_{ij} - \nabla_0(g^{ij}v_{ij}) + v^{\alpha 0}\nabla_\alpha f\Big)\Big]e^{-f}\,\mathrm{dA},\end{aligned}$$

where the derivative is to be taken on $(g,f) \in \mathrm{met}(M) \times C^\infty(M)$*,* $\frac{\partial}{\partial t} g(t) = v$ *and* $\frac{\partial}{\partial t} f(t) = h$*. If, in addition,* $\frac{v^\alpha{}_\alpha}{2} - h = 0$ *on* M*, then the following equality holds:*

$$\delta\mathcal{F}(v_{\alpha\beta},h) = -\int_M v^{\alpha\beta}(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f)e^{-f}\,\mathrm{dV} - \int_{\partial M}\Big(\widehat{\nabla}_i v^{i0} - Hv^0{}_0 + \mathcal{A}^{ij}v_{ij} - \nabla_0(g^{ij}v_{ij}) + v^{\alpha 0}\nabla_\alpha f\Big)e^{-f}\,\mathrm{dA}\,.$$

Remark 1.10. Here, it is important to observe that $\frac{v^\alpha{}_\alpha}{2} - h$ vanishes identically on M if and only if the measure $\mathrm{d}m = e^{-f}\,\mathrm{dV}$ remains fixed on M, since $\delta(e^{-f}\,\mathrm{dV}) = (\frac{v^\alpha{}_\alpha}{2} - h)e^{-f}\,\mathrm{dV}$.

Remark 1.11. For the sake of standard notation, we observe that $v_{00} := v^\zeta{}_0 g_{\zeta 0} = v^0{}_0$.

Proof. The *variation* of $\mathcal{F}$ in direction $(v,h) \in S^2(M) \times C^\infty(M)$, is defined to be

$$\delta\mathcal{F}(v,h)(g,f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{F}(g+tv, f+th),$$

where $S^2(M)$ is the set of all symmetric 2-tensor fields on M. For simplicity, let us omit the duple (g,f). This implies that

$$\delta\mathcal{F}(v,h) = \int_M \Big[\nabla^\alpha\nabla^\beta v_{\alpha\beta} - \nabla^\beta\nabla_\beta v^\alpha{}_\alpha - v^{\alpha\beta}R_{\alpha\beta} - v^{\alpha\beta}\nabla_\alpha f\nabla_\beta f + 2g(\nabla f, \nabla h) + (R_g + |\nabla f|^2)\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)\Big]e^{-f}\,\mathrm{dV}\,.$$

We must compute the integrals on the right-hand side of the previous identity. We start using Green's formula (1.6) to obtain

$$\int_M e^{-f}(-\Delta_g v^\alpha{}_\alpha)\,\mathrm{dV} = -\int_M \Delta_g e^{-f} v^\alpha{}_\alpha\,\mathrm{dV} - \int_{\partial M} v^\alpha{}_\alpha e_0 e^{-f}\,\mathrm{dA} + \int_{\partial M} e^{-f} e_0 v^\alpha{}_\alpha\,\mathrm{dA}\,.$$

Now, we use integration by parts (1.5) to compute

$$\begin{aligned}
\int_M e^{-f}\nabla_\alpha\nabla_\beta v^{\alpha\beta}\,\mathrm{dV} &= -\int_M \nabla_\alpha e^{-f}\nabla_\beta v^{\alpha\beta}\,\mathrm{dV} + \int_M \nabla_\alpha(e^{-f}\nabla_\beta v^{\beta\alpha})\,\mathrm{dV}\\
&= -\int_M \nabla_\alpha e^{-f}\nabla_\beta v^{\alpha\beta}\,\mathrm{dV} - \int_{\partial M}\nabla_\beta v^{\beta 0}e^{-f}\,\mathrm{dA}\\
&= \int_M \nabla_\alpha\nabla_\beta e^{-f} v^{\alpha\beta}\,\mathrm{dV} - \int_M \nabla_\beta(\nabla_\alpha e^{-f} v^{\alpha\beta})\,\mathrm{dV} - \int_{\partial M}\nabla_\beta v^{\beta 0}e^{-f}\,\mathrm{dA}\\
&= \int_M \nabla_\alpha\nabla_\beta e^{-f} v^{\alpha\beta}\,\mathrm{dV} + \int_{\partial M}\nabla_\alpha e^{-f} v^{\alpha 0}\,\mathrm{dA} - \int_{\partial M}\nabla_\beta v^{\beta 0}e^{-f}\,\mathrm{dA}\,.
\end{aligned}$$

Notice that $\nabla_\alpha\nabla_\beta e^{-f} = -e^{-f}\nabla_\alpha\nabla_\beta f + e^{-f}\nabla_\alpha f\nabla_\beta f$. Finally by means of (1.5), we have

$$2\int_M e^{-f}g(\nabla f, \nabla h)\,\mathrm{dV} = -2\int_M g(\nabla e^{-f}, \nabla h)\,\mathrm{dV} = 2\int_M \big(\Delta_g e^{-f}\big)h\,\mathrm{dV} + 2\int_{\partial M} he_0\big(e^{-f}\big)\,\mathrm{dA}\,.$$

Thus

$$\delta\mathcal{F}(v,h) = -\int_M v^{\alpha\beta}(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f)e^{-f}\,\mathrm{dV} + \int_M \Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)(R_g - |\nabla f|^2 + 2\Delta_g f)e^{-f}\,\mathrm{dV}$$
$$+2\int_{\partial M}\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)e_0 f e^{-f}\,\mathrm{dA} - \int_{\partial M}\Big(\nabla_\alpha v^{\alpha 0} - \nabla_0(v^\alpha{}_\alpha) + v^{\alpha 0}\nabla_\alpha f\Big)e^{-f}\,\mathrm{dA}\,.$$

By (1.2), we get $\nabla_\alpha v^{\alpha 0} - \nabla_0(v^\alpha{}_\alpha) = \nabla_i v^{i0} - \nabla_0 v^i_i = \widehat{\nabla}_i v^{i0} - Hv^0{}_0 + \mathcal{A}^{ij}v_{ij} - \nabla_0(g^{ij}v_{ij})$. This is enough to conclude the proof of the proposition. □

Remark 1.12. If M has no boundary then Proposition 1.9 appears in [Per02, Sect. 1.1]. In addition, if $\frac{v^\alpha{}_\alpha}{2} - h = 0$ on M, then

$$\delta\mathcal{F}(v,h) = -\int_M v^{\alpha\beta}(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f)e^{-f}\,\mathrm{dV}\,. \tag{1.21}$$

One can think that $\mathcal{F}$ has a gradient-like structure constraint to

$$\mathscr{C} := \Big\{v\in S^2(M) : v = \frac{\partial}{\partial t}g(t) \text{ and } f = \ln(\frac{\mathrm{dV}}{\mathrm{d}m})\Big\}.$$

If $\frac{v^\alpha{}_\alpha}{2} - h = 0$ on M, then $f = \ln(\frac{\mathrm{dV}}{\mathrm{d}m})$, see Remark 1.10. As pointed out by Perelman, the *gradient structure* for Ricci flow comes from the following functional

$$\mathcal{F}^m(g) = \mathcal{F}\Big(g, \ln\big(\frac{\mathrm{dV}}{\mathrm{d}m}\big)\Big) = \int_M (R_g + |\nabla f|^2)\,\mathrm{d}m.$$

Now we introduce the *weighted inner product* on $S^2(M)$ as

$$\langle v,s\rangle := \int_M v^{\gamma\zeta}s_{\gamma\zeta}e^{-f}\,\mathrm{dV},$$

so that we can define the *gradient* of $\mathcal{F}^m$ at g given by

$$\delta\mathcal{F}^m(g)(v) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{F}^m(g+tv) = \langle \operatorname{grad}\mathcal{F}^m(g), v\rangle,$$

for all $v\in\mathscr{C}$. This gradient-like structure was the motivation for Perelman to consider the system below

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{g}(t) := 2\operatorname{grad}\mathcal{F}^m(\widetilde{g}(t)) = -2(\mathrm{Ric}_{\widetilde{g}(t)} + \mathrm{Hess}_{\widetilde{g}(t)}\widetilde{f}(t)), \\ \frac{\partial}{\partial t}\widetilde{f}(t) := h = \frac{v^\alpha{}_\alpha}{2} = -\Delta_{\widetilde{g}(t)}\widetilde{f}(t) - R_{\widetilde{g}(t)}. \end{cases} \tag{1.22}$$

To find a solution to (1.22) we consider a solution of the *backward heat equation* $\frac{\partial}{\partial t}f(t) = -\Delta_{g(t)}f(t) + |\nabla_{g(t)}f(t)|^2 - R_{g(t)}$ along the Ricci flow in $M\times[a,b]$, which is obtained as follows. Let $[a,b]$ be a sub-interval of $[0,T)$ and $g(t)$ satisfying the Ricci flow equation $\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}_{g(t)}$

in $[a,b]$. Take $z(t) := e^{-f(t)}$ and define $s = T - t$. Since $\Delta_g z = (|\nabla f|^2 - \Delta_g f)z$, one has

$$\frac{\partial}{\partial s}z = \frac{\partial}{\partial t}z\frac{\partial}{\partial s}t = e^{-f}\frac{\partial}{\partial t}f = e^{-f}(-\Delta_g f + |\nabla_g f|^2 - R_g) = \Delta_g z - R_g z$$

which is a parabolic equation in $M \times [a,b]$. It guarantees the existence of $f(t)$ along the Ricci flow in $M \times [a,b]$. Now, let $\{\phi_t\}_{t\in[a,b]}$ be the one-parameter family of diffeomorphisms generated by $\{-\nabla_{g(t)} f(t)\}_{t\in[a,b]}$, with $\phi_a = \mathrm{Id}$. By setting $\widetilde{g}(t) := \phi_t^* g(t)$ and $\widetilde{f}(t) := \phi_t^* f(t)$, we have

$$\frac{\partial}{\partial t}\widetilde{g}(t) = \phi_t^*(\frac{\partial}{\partial t}g(t)) + \phi_t^*\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t}g(t) = -2(\mathrm{Ric}_{\widetilde{g}(t)} + \mathrm{Hess}_{\widetilde{g}(t)}\widetilde{f}(t))$$

and

$$\frac{\partial}{\partial t}\widetilde{f}(t) = \phi_t^*(\frac{\partial}{\partial t}f(t)) + \phi_t^*\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t}f(t) = -\Delta_{\widetilde{g}(t)}\widetilde{f}(t) - R_{\widetilde{g}(t)}.$$

The two latter equations imply $\frac{\partial}{\partial t}\widetilde{f} = \frac{\widetilde{g}^{\alpha\beta}\frac{\partial}{\partial t}\widetilde{g}_{\alpha\beta}}{2}$. Hence, $(\widetilde{g}(t), \widetilde{f}(t))$ is a solution to (1.22).

Now, note that we can use (1.21) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\Big(\frac{\partial}{\partial t}\widetilde{g}, \frac{\partial}{\partial t}\widetilde{f}\Big) = 2\int_M |\mathrm{Ric}_{\widetilde{g}} + \mathrm{Hess}_{\widetilde{g}}\widetilde{f}|^2 e^{-\widetilde{f}}\,\mathrm{dV}_{\widetilde{g}}.$$

So, $\mathcal{F}$ is constant in t if and only if $\big(\widetilde{g}(t), \widetilde{f}(t)\big)$ is a gradient steady Ricci soliton on M. System (1.22) is known as *Perelman's modified Ricci flow* in $M \times [a,b]$.

We say that g is a *critical point* of $\mathcal{F}^m$ if $\delta\mathcal{F}^m(v) = 0$ for all $v \in \mathscr{C}$. Note that g is a critical point of $\mathcal{F}^m$ constraint to $\mathscr{C}$ if and only if the orthogonal projection of $\operatorname{grad}\mathcal{F}^m(g) = -\mathrm{Ric}_g - \mathrm{Hess}_g f$ onto $\mathscr{C}$ is null. Then, the gradient steady Ricci solitons are critical points of $\mathcal{F}^m$ constraint to $\mathscr{C}$.

In [Per02], the functional

$$\mathcal{W}(g,f,\tau) = \int_M [\tau R_\infty + f - n]u\,\mathrm{dV} = \int_M [\tau(R + |\nabla f|^2) + f - n]u\,\mathrm{dV}, \tag{1.23}$$

for $\tau > 0$ and smooth functions f on M it is also considered, where

$$u := \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}.$$

An associated entropy is defined by

$$\mu(g,\tau) = \inf\Big\{\mathcal{W}(g,f,\tau), \int_M u\,\mathrm{dV} = 1\Big\}.$$

In the same way as in (1.22), Perelman [Per02, Sect. 3.1] also showed that the system

$$\begin{cases} \frac{\partial}{\partial t} g = -2(\mathrm{Ric}_g + \mathrm{Hess}_g f), \\ \frac{\partial}{\partial t} f = -\Delta_g f - R_g + \frac{n}{2\tau}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\tau = -1, \end{cases} \tag{1.24}$$

has a solution in $M \times [a,b]$. Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}\Big(\frac{\partial}{\partial t}g, \frac{\partial}{\partial t}f, \frac{\mathrm{d}}{\mathrm{d}t}\tau\Big) = 2\int_M \tau |\,\mathrm{Ric}_g + \mathrm{Hess}_g f - \frac{1}{2\tau}g|^2 u \,\mathrm{dV}_g,$$

where $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$.

We highlight that $\mathcal{F}$ is nondecreasing in time under Perelman's modified Ricci flow (1.22) while $\mathcal{W}$ is nondecreasing in time under system (1.24).

An important consequence of this entropy formula is a lower volume ratio bound for solutions of the Ricci flow on a finite time interval $[0,T)$, asserting the existence of a constant $\kappa > 0$, only depending on n, T and $g(0)$, such that the inequality

$$\frac{\mathrm{Vol}_t(B_r^t(x_0))}{r^{n+1}} \geqslant \kappa$$

holds for all $t \in [0,T)$ and $r \in [0,\sqrt{T})$ for balls $B_r^t(x_0)$ (with respect to $g(t)$) in which the inequality $r^2|\,\mathrm{Rm}\,| \leqslant 1$ for the Riemann tensor of $g(t)$ is satisfied.

In [CM19], Cortissoz and Murcia established the monotonicity of $\mathcal{W}$ and $\mathcal{F}$-Perelman's functionals on surfaces M^2 with boundary ∂M under evolution equations given by

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} & \text{in} \quad M^2 \times (0,T), \\ k_{g(t)}(\,\cdot\,,t) = \phi(\,\cdot\,) & \text{on} \quad \partial M \times (0,T), \\ \frac{\partial}{\partial t} f(t) = -\Delta_{g(t)} f(t) + |\nabla_{(t)} f(t)|^2 - R_{g(t)} + \frac{n}{2\tau} & \text{in} \quad M^2 \times (0,T), \\ e_0 f(t) = 0 & \text{on} \quad \partial M \times (0,T), \\ \frac{\mathrm{d}}{\mathrm{d}t}\tau = -1 & \text{in} \quad M^2 \times (0,T), \end{cases}$$

where k_g is the geodesic curvature of ∂M, both with respect to the time evolving metric g, and ϕ is a smooth real valued function, which is constant in space, defined on $\partial M \times [0,\infty)$ and which satisfies the compatibility condition $\phi(\cdot,0) = k_{g_0}$ (disregard τ for the case of $\mathcal{F}$).

In [Eck07], $\mathcal{W}$-Perelman's Entropy (1.23) had been studied by Ecker in the setting of bounded domains in Euclidean space $\mathbb{R}^n$ equipped with its standard metric g_0, whose boundary evolves by mean curvature flow. More precisely, he adapted Perelman's entropy formula to the situation where a family of bounded domains $\{\Omega_t\}_{t\in[0,T)}$ in $\mathbb{R}^n$ with smooth boundary hypersurfaces family $\{\partial\Omega_t\}_{t\in[0,T)}$ is evolving with smooth normal speed

$$\beta_{\partial\Omega_t} = -\frac{\partial}{\partial t}x \cdot \nu,$$

where x denotes the embedding map from $\partial\Omega_t$ to $\mathbb{R}^n$, and ν stands for the outward unit normal vector field along $\partial\Omega_t$. For bounded domains $\Omega \subset \mathbb{R}^n$, $\tau > 0$, smooth functions $f : \overline{\Omega} \to \mathbb{R}$ and $\beta : \partial\Omega \to \mathbb{R}$, Ecker considered the functional

$$\mathcal{W}_\beta(\Omega, f, \tau) = \int_\Omega [\tau|\nabla f|^2 + f - n]u\,\mathrm{d}x + 2\tau\int_{\partial\Omega}\beta u\,\mathrm{dA} \tag{1.25}$$

and the associated entropy

$$\mu_\beta(\Omega,\tau) = \inf\left\{\mathcal{W}_\beta(\Omega,f,\tau), \int_\Omega u\,\mathrm{d}x = 1 \text{ where } u := \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\right\}.$$

Suppose that $\{\Omega_t\}$ evolves as above, $\tau(t) > 0$, $\frac{\mathrm{d}}{\mathrm{d}t}\tau = -1$, and f satisfies the evolution equation

$$\frac{\partial}{\partial t}f = -\Delta_g f + |\nabla f|^2 + \frac{n}{2\tau}$$

in Ω_t with Neumann boundary condition $\nabla f \cdot \nu = \beta$ on $\partial\Omega_t$. If $\{\phi_t\}_{t\in[0,T)}$ is an one-parameter family of diffeomorphisms $\phi_t : \overline{\Omega} \to \overline{\Omega}_t$ generated by $\{-\nabla f(x,t)\}_{t\in[0,T)}$ with $x = \phi_t(q)$, $q \in \overline{\Omega}$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) = 2\tau\int_{\Omega_t} &|\operatorname{Hess} f - \frac{1}{2\tau}g_0|^2 u\,\mathrm{d}x \\ &+ 2\tau\int_{\partial\Omega_t}\Big(\frac{\partial}{\partial t}\beta - 2\langle\widehat{\nabla}\beta, \widehat{\nabla}f\rangle + \mathcal{A}(\widehat{\nabla}f, \widehat{\nabla}f) - \frac{\beta}{2\tau}\Big)u\,\mathrm{dA}.\end{aligned}$$

In the important case of mean curvature flow, that is, where $\beta_{\partial\Omega_t}$ is the mean curvature H of the hypersurfaces $\partial\Omega_t$, the expression

$$\mathcal{Z}(\widehat{\nabla}f) := \frac{\partial}{\partial t}H - 2\langle\widehat{\nabla}H, \widehat{\nabla}f\rangle + \mathcal{A}(\widehat{\nabla}f, \widehat{\nabla}f)$$

is the central quantity in Hamilton's Harnack inequality for convex solutions of the mean curvature flow (see [Ham95]).

Ecker conjectured that $\mathcal{W}_\beta$-functional for Ω is nondecreasing in time under the mean curvature flow of any compact hypersurface in $\mathbb{R}^n$. More precisely

Conjecture 1.13. *In the case of mean curvature flow in $\mathbb{R}^n$ for compact embedded hypersurfaces $\partial\Omega_t$, it is expected that*

$$\int_{\Omega_t}\big|\operatorname{Hess} f - \frac{1}{2\tau}g_0\big|^2 u\,\mathrm{d}x + \int_{\partial\Omega_t}\big(\mathcal{Z}(\widehat{\nabla}f) - \frac{H}{2\tau}\big)u\,\mathrm{dA} \geqslant 0.$$

To the author's knowledge, this conjecture is still open. Lott was inspired by Ecker's work to define a weighted version of the following action (cf. [Lot12])

$$I_{GHY}(g) = \int_M R_g \,\mathrm{dV} + 2\int_{\partial M} H \,\mathrm{dA}, \tag{1.26}$$

If $n = 2$, then by Gauss-Bonnet theorem, $I_{GHY}(g) = 4\pi\chi(M)$. This action appears previously in the works by Gibbons and Hawking [GH77], York [Yor72], and later by Araújo [Ara03]. I_{GHY} is known as *Gibbons-Hawking-York (GHY, for short) action*. Now we shall briefly mention some of their results. Gibbons and Hawking used this approach to evaluate the entropy of the actions of the Kerr-Newman solutions and de Sitter space and found that it is always equal to one quarter the area of the event horizon in fundamental units. Moreover, in the case of a stationary system such as a star with no event horizon, the gravitational field has no entropy.

York gave results concerning the action principle, choice of canonical variables, and initial-value equations strengthen this identification. One of the new canonical variables is shown to play the role of "time" in the formalism.

Araújo characterized the critical points of I_{GHY} restricted to spaces of Riemannian metrics satisfying various volume and area constraints when the dimension of the manifold is bigger than three. In addition, he computed the second variation of I_{GHY} at critical points and provided directions in which it is positive, negative or zero. These results generalize to manifolds with boundary some well-known results that hold in the case of manifolds without boundary.

1.2.1 Evolution of the weighted total mean curvature functional

Lott defined a version of Perelman's $\mathcal{F}$-funcional for manifolds with boundary that can be considered as a weighted version of the GHY-action on $\mathrm{met}(M) \times C^\infty(M)$ as follows

$$I_\infty(g, f) = \int_M R_\infty e^{-f} \,\mathrm{dV} + 2\int_{\partial M} H_\infty e^{-f} \,\mathrm{dA}, \tag{1.27}$$

and called it *weighted GHY-action* I_∞ (see [Lot12]).

Before studying the evolution of I_∞, we are going to establish the evolution of the mean curvature at g, and then of the *weighted total mean curvature functional* (see Lemma 1.14). We know three ways to do this. In the first one, the unit normal on ∂M is written in terms of the family of metrics on M to express the derivative of the second fundamental form on ∂M (see [Ara03, p. 89]). In the second one, the unit normal on ∂M is used locally as a gradient of some smooth function on M (see [GM19, Lem. 2]). The last, we will make it here in the same way as in [Mia03, Lem. 1].

Let $\{g(t)\}$ be a family of metrics on M such that $g(0) = g$ and $\frac{\partial}{\partial t}\big|_{t=0} g(t) = v$. For each t, we denote ∇^t as the covariant derivative with respect to $g(t)$ with $\nabla := \nabla^0$. We define $e(t)$ to be the inward unit normal vector field on $(\partial M, g(t))$. We also choose $\{x_\alpha\}_{\alpha=0}^{n-1}$ to be a local coordinate chart for M such that $\{x_i\}_{i=1}^{n-1}$ gives a local chart for ∂M and ∂_0 coincides with $e_0 := e(0)$. Then by definition

$$\mathcal{A}_{ij}(g(t)) = g(t)\Big(\nabla^t_{\partial_i}\partial_j, e(t)\Big),$$

where $\mathcal{A}_{ij}(g(t))$ stands for the second fundamental form of $(\partial M, g(t))$. Recall that $\alpha, \beta, \dots$ run through $\{0, 1, \dots, n-1\}$ and $i, j, \dots$ run through $\{1, 2, \dots, n-1\}$. Now we are going to calculate the evolution of the second fundamental form. First observe that

$$\frac{\partial}{\partial t} g(t)\big(X(t), Y(t)\big)\Big|_{t=0} = v(X,Y) + g\big(\frac{\partial}{\partial t}X(t)\Big|_{t=0}, Y\big) + g\big(X, \frac{\partial}{\partial t}Y(t)\Big|_{t=0}\big)$$

for all $X(t), Y(t)$ vector field on M. So

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{A}_{ij}(g(t))\Big|_{t=0} &= v(\nabla_{\partial_i}\partial_j, e_0) + g\Big(\frac{\partial}{\partial t}(\nabla^t_{\partial_i}\partial_j)\Big|_{t=0}, e_0\Big) + g\Big(\nabla_{\partial_i}\partial_j, \frac{\partial}{\partial t}e(t)\Big|_{t=0}\Big) \\
&= \Gamma^{\alpha}_{ij} v_{\alpha 0} + g\Big(\frac{\partial}{\partial t}(\nabla^t_{\partial_i}\partial_j)\Big|_{t=0}, e_0\Big) + \Gamma^{\alpha}_{ij} g\Big(\partial_\alpha, \frac{\partial}{\partial t}e(t)\Big|_{t=0}\Big) \\
&= \Gamma^{k}_{ij} v_{k0} + \Gamma^{0}_{ij} v_{00} + g\Big(\frac{\partial}{\partial t}(\nabla^t_{\partial_i}\partial_j)\Big|_{t=0}, e_0\Big) + \Gamma^{k}_{ij} g\Big(\partial_k, \frac{\partial}{\partial t}e(t)\Big|_{t=0}\Big) \\
&\quad + \Gamma^{0}_{ij} g\Big(e_0, \frac{\partial}{\partial t}e(t)\Big|_{t=0}\Big).
\end{aligned}$$

The facts that $g(t)(e(t), \partial_k) = 0$ and $g(t)(e(t), e(t)) = 1$ imply

$$\begin{cases} v_{0k} + g\Big(\partial_k, \frac{\partial}{\partial t}e(t)\Big|_{t=0}\Big) = 0, \\ v_{00} + 2g\Big(e_0, \frac{\partial}{\partial t}e(t)\Big|_{t=0}\Big) = 0. \end{cases}$$

Since

$$g\Big(\frac{\partial}{\partial t}(\nabla^t_{\partial_i}\partial_j)\Big|_{t=0}, e_0\Big) = \frac{1}{2}(\nabla_i v^0{}_j + \nabla_j v^0{}_i - \nabla_0 v_{ij})$$

we conclude that

$$\frac{\partial}{\partial t}\mathcal{A}_{ij}(g(t))\Big|_{t=0} = \frac{1}{2}(\nabla_i v^0{}_j + \nabla_j v^0{}_i - \nabla_0 v_{ij}) + \frac{1}{2}v_{00}\mathcal{A}_{ij}. \tag{1.28}$$

This finishes the evolution of the second fundamental form. Now, we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}H(g(t))\Big|_{t=0} &= \frac{\partial}{\partial t}g(t)^{ij}\Big|_{t=0}\mathcal{A}_{ij} + g^{ij}\frac{\partial}{\partial t}\mathcal{A}_{ij}(g(t))\Big|_{t=0} \\
&= -v^{ij}\mathcal{A}_{ij} + \frac{1}{2}g^{ij}(\nabla_i v^0{}_j + \nabla_j v^0{}_i - \nabla_0 v_{ij}) + \frac{1}{2}v_{00}H,
\end{aligned} \tag{1.29}$$

and observe that (1.2) implies

$$\frac{\partial}{\partial t}H(g(t))\Big|_{t=0} = -v^{ij}\mathcal{A}_{ij} + \frac{1}{2}g^{ij}(\widehat{\nabla}_i v^0{}_j - \mathcal{A}_{ij}v^0{}_0 + g^{kl}\mathcal{A}_{il}v_{jk} + \widehat{\nabla}_j v^0{}_i - \mathcal{A}_{ji}v^0{}_0$$

$$
\begin{aligned}
&+g^{kl}\mathcal{A}_{jl}v_{ik}-\nabla_0 v_{ij}\big)+\frac{1}{2}v_{00}H \\
&=\widehat{\nabla}_i v^{i0}-\frac{1}{2}\Big(g^{ij}\nabla_0 v_{ij}+v_{00}H\Big).
\end{aligned}
\tag{1.30}
$$

This finishes the evolution of the mean curvature.

Now we calculate the evolution of the weighted total mean curvature functional.

Lemma 1.14. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M. Consider the weighted total mean curvature functional $\mathscr{L}:\mathrm{met}(M)\times C^\infty(M)\to\mathbb{R}$ defined as*

$$
\mathscr{L}(g,f)=2\int_{\partial M} He^{-f}\,\mathrm{dA}\,.
$$

Then, the following equality holds:

$$
\delta\mathscr{L}(v,h)=\int_{\partial M}\Big[2\widehat{\nabla}_i v^{i0}-g^{ij}\nabla_0 v_{ij}-v_{00}H+2H\Big(\frac{g^{ij}v_{ij}}{2}-h\Big)\Big]e^{-f}\,\mathrm{dA}\,.
$$

In particular, if $\frac{v^\alpha{}_\alpha}{2}-h=0$ on M, then it reduces to

$$
\delta\mathscr{L}(v,h)=\int_{\partial M}\big(2\widehat{\nabla}_i v^{i0}-g^{ij}\nabla_0 v_{ij}-(v_{00}+v^{00})H\big)e^{-f}\,\mathrm{dA}\,.
$$

Proof. The *variation* of $\mathscr{L}$ in direction $(v,h)\in S^2(M)\times C^\infty(M)$, is defined to be

$$
\delta\mathscr{L}(v,h)(g,f)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathscr{L}(g+tv,f+th),
$$

Thus, by equation (1.30), we have

$$
\begin{aligned}
\delta\mathscr{L}(v,h)&=2\int_{\partial M}\delta\{H\}e^{-f}\,\mathrm{dA}+2\int_{\partial M}H\delta\{e^{-f}\,\mathrm{dA}\}\\
&=2\int_{\partial M}\Big(\widehat{\nabla}_i v^{i0}-\frac{1}{2}\big(g^{ij}\nabla_0 v_{ij}+v_{00}H\big)\Big)e^{-f}\,\mathrm{dA}+2\int_{\partial M}H\Big(\frac{g^{ij}v_{ij}}{2}-h\Big)e^{-f}\,\mathrm{dA}
\end{aligned}
$$

which is the first part of the lemma. Now, note that $g^{ij}v_{ij}=\mathrm{tr}_g(v|_{\partial M})=v^\alpha{}_\alpha-v_{00}$, and as

$$
v_{00}=v(e_0,e_0)=v(g^{\alpha\beta}g_{\alpha0}\partial_\beta,g^{\gamma\xi}g_{\gamma0}\partial_\xi)=g^{\alpha\beta}g^{\gamma\xi}g_{\alpha0}g_{\gamma0}v_{\beta\xi}=g^{0\beta}g^{0\xi}v_{\beta\xi}=v^{00}
$$

we obtain $v^\alpha{}_\alpha=g^{ij}v_{ij}+v^{00}$. In particular, if $\frac{v^\alpha{}_\alpha}{2}-h=0$ on M, then

$$
\frac{g^{ij}v_{ij}}{2}-h=\frac{v^\alpha{}_\alpha}{2}-h-\frac{1}{2}v^{00}=-\frac{1}{2}v^{00}.
$$

Thus

$$
\delta\mathscr{L}(v,h)=\int_{\partial M}\Big(2\widehat{\nabla}_i v^{i0}-g^{ij}\nabla_0 v_{ij}-v_{00}H\Big)e^{-f}\,\mathrm{dA}-\int_{\partial M}Hv^{00}e^{-f}\,\mathrm{dA}\,.
$$

This finishes the proof. $\square$

1.2.2 Evolution of the weighted GHY-action

We start rewriting the weighted GHY-action as follows

$$I_\infty(g,f) = \int_M (R_g + |\nabla f|^2) e^{-f}\, \mathrm{dV} + 2\int_{\partial M} H e^{-f}\, \mathrm{dA}\,. \tag{1.31}$$

It follows by using $\Delta_g e^{-f} = (|\nabla f|^2 - \Delta_g f)e^{-f}$ and divergence theorem in (1.27).

Lott computed the evolution of I_∞ directly from (1.27). From Proposition 1.9 and Lemma 1.14, we give proof of this evolution, as follows.

Proposition 1.15. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M. Let I_∞ be the weighted GHY-action on* $\mathrm{met}(M)\times C^\infty(M)$ *defined as in* (1.27)*. Then its evolution is given by*

$$\begin{aligned}\delta I_\infty(v,h) = &\int_M \Big[-v^{\alpha\beta}(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f) + \Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)(R_g - |\nabla f|^2 + 2\Delta_g f)\Big] e^{-f}\,\mathrm{dV}\\ &+ \int_{\partial M}\Big[2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)e_0 f - \mathcal{A}^{ij}v_{ij} - v^{00}e_0 f + 2H\Big(\frac{g^{ij}v_{ij}}{2} - h\Big)\Big]e^{-f}\,\mathrm{dA}\,.\end{aligned}$$

Proof. For the sake of convenience, we are following a different way to address the problem, however, the main ideas are the same as in Lott. For it, we are working with (1.31) instead of (1.27) so that

$$I_\infty(g,f) = \mathcal{F}(g,f) + \mathscr{L}(g,f).$$

Thus,

$$\delta I_\infty(v,h) = \delta\mathcal{F}(v,h) + \delta\mathscr{L}(v,h).$$

The first term on the right-hand side is provided by Proposition 1.9, and the second by Lemma 1.14. From which, one has

$$\begin{aligned}\delta I_\infty(v,h) = &\int_M \Big[-v^{\alpha\beta}(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f) + \Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)(R_g - |\nabla f|^2 + 2\Delta_g f)\Big] e^{-f}\,\mathrm{dV}\\ &+ \int_{\partial M}\Big[2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)e_0 f - \Big(\widehat{\nabla}_i v^{i0} - Hv^0{}_0 + \mathcal{A}^{ij}v_{ij} - \nabla_0(g^{ij}v_{ij}) + v^{\alpha 0}\nabla_\alpha f\Big)\Big]e^{-f}\,\mathrm{dA}\\ &+ \int_{\partial M}\Big[2\widehat{\nabla}_i v^{i0} - g^{ij}\nabla_0 v_{ij} - v_{00}H + 2H\Big(\frac{g^{ij}v_{ij}}{2} - h\Big)\Big]e^{-f}\,\mathrm{dA}\\ = &\int_M \Big[-v^{\alpha\beta}(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f) + \Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)(R_g - |\nabla f|^2 + 2\Delta_g f)\Big] e^{-f}\,\mathrm{dV}\\ &+ \int_{\partial M}\Big[2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)e_0 f - \mathcal{A}^{ij}v_{ij} + \widehat{\nabla}_i v^{i0} - v^{i0}\widehat{\nabla}_i f - v^{00}e_0 f + 2H\Big(\frac{g^{ij}v_{ij}}{2} - h\Big)\Big]e^{-f}\mathrm{dA}\,.\end{aligned}$$

The proposition follows from $\widehat{\nabla}_i v^{i0} e^{-f} = \widehat{\nabla}_i (v^{i0} e^{-f}) + e^{-f} \widehat{\nabla}_i f v^{i0}$ and divergence theorem. $\square$

Corollary 1.16 ([Lot12, Prop. 2])**.** *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M. Let I_∞ be the weighted GHY-action on* $\operatorname{met}(M) \times C^\infty(M)$ *defined as in* (1.27)*. If $\frac{v^\alpha{}_\alpha}{2} - h = 0$ on M, then*

$$\delta I_\infty(v,h) = -\int_M v^{\alpha\beta}(R_{\alpha\beta} + \nabla_\alpha \nabla_\beta f) e^{-f}\, \mathrm{dV} - \int_{\partial M} \Big(v^{ij} \mathcal{A}_{ij} + v^{00}(H + e_0 f) \Big) e^{-f}\, \mathrm{dA}\,.$$

By taking f constant in Proposition 1.15, we obtain.

Corollary 1.17 ([Ara03, Sect. 2])**.** *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M. Let I_{GHY} be the GHY-action on* $\operatorname{met}(M)$ *defined as in* (1.26)*. Then*

$$\delta I_{GHY}(v) = -\int_M v^{\alpha\beta}\Big(R_{\alpha\beta} - \frac{R_g}{2} g_{\alpha\beta}\Big)\, \mathrm{dV} - \int_{\partial M} v^{ij}(\mathcal{A}_{ij} - H g_{ij})\, \mathrm{dA}\,. \tag{1.32}$$

We are in a position to analyze the critical metrics of I_{GHY}. Recall that $g \in \operatorname{met}(M)$ is a *critical point* (or *critical metric*) of I_{GHY} if $\delta I_{GHY}(v) = 0$ for all $v = \frac{\partial}{\partial t} g(t)$.

Corollary 1.18 ([Lot12, Prop. 2])**.** *Let I_{GHY} be the GHY-action on* $\operatorname{met}(M)$ *defined in* (1.26)*, with $n \geqslant 3$. If it is fixed an induced metric $g_{\partial M}$ on ∂M, then the critical points of I_{GHY} are the Ricci-flat metrics on M. On the other hand, if it is considered all variations, then the critical points are the Ricci flat metrics on M with the totally geodesic boundary ∂M.*

Proof. Assume that the induced metric $g_{\partial M}$ is fixed. Then $v_{ij} = 0$ on ∂M and by (1.32) we have

$$\int_M \big\langle v, \mathrm{Ric} - \frac{R_g}{2} g \big\rangle\, \mathrm{dV} = \int_M v^{\alpha\beta}\Big(R_{\alpha\beta} - \frac{R_g}{2} g_{\alpha\beta}\Big)\, \mathrm{dV} = 0,$$

for all $v \in S^2(M)$. This implies $\mathrm{Ric}_g - \frac{R_g}{2} g = 0$ on M. Take the traces in this equation to obtain $R_g(1 - \frac{n}{2}) = 0$, so $R_g = 0$. Therefore, g is Ricci flat. This proves the first statement. In the general case, we have

$$\int_M \big\langle v, \mathrm{Ric} - \frac{R_g}{2} g \big\rangle\, \mathrm{dV} + \int_{\partial M} \big\langle v, \mathcal{A} - H g \big\rangle\, \mathrm{dA} = 0,$$

for all $v \in S^2(M)$ from which we obtain that the critical points are Ricci flat metrics on M with totally geodesic boundary ∂M. $\square$

Araújo classified the critical points of I_{GHY} constraint to set $\mathrm{met_{a,b}(M)} = \{\mathrm{g} \in \mathrm{met(M)} : \mathrm{a\,Vol_g(M) + b\,Area_g(\partial M) = 1}\}$ where $\mathrm{Area}_g(\partial M)$ is the area of $(\partial M, g)$, $\mathrm{Vol}_g(M)$ is the volume of (M, g) and a, b are real numbers with either $a > 0$ or $a = 0$, $b = 1$. He showed that when

$a=1$ and $b=0$, the critical points of the functional I_{GHY} correspond to Einstein metrics with totally geodesic boundary. When $a=0$ and $b=1$, the critical points correspond to Ricci flat metrics with umbilic boundary of constant mean curvature (cf. [Ara03, Cors. 2.2 and 2.3]).

Remark 1.19 ([Lot12, Sect. 3.2])**.** The variations in Corollary 1.16 all fix the measure $e^{-f}\,\mathrm{dV}$. If we also fix an induced metric $g_{\partial M}$ on ∂M then the critical points of I_∞ are gradient steady Ricci solitons on M that satisfy $H+e_0 f=0$ on ∂M. On the other hand, if we allow variations that do not fix the boundary metric then the critical points are gradient steady Ricci solitons on M with totally geodesic boundary and for which f satisfies Neumann boundary conditions.

From this remark, one can obtain an idea of how Lott motivated his boundary conditions to the backward heat equation (1.22). In Chapter 2, we will follow the Lott's program.

1.2.3 Evolution of the $\mathcal{W}_\infty$-type entropy functional

In [Lot12, Example 2], Lott pointed out that after making the change from $\mathcal{F}$-type functional (1.27) to $\mathcal{W}$-type functional (1.25), it is possible studying its evolution. Here, we address this issue in more detail. For it, we define $\mathcal{W}_\infty$*-type entropy* as follows

$$\mathcal{W}_\infty(g,f,\tau)=\int_M\big(\tau R_\infty+f-n\big)u\,\mathrm{dV}+2\int_{\partial M}\tau H_\infty u\,\mathrm{dA}\,. \tag{1.33}$$

where $u:=(4\pi\tau)^{-\frac{n}{2}}e^{-f}$. Using Proposition 1.15, we have the following evolution of $\mathcal{W}_\infty$.

Proposition 1.20. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and consider the* $\mathcal{W}_\infty$-type entropy *on* $\mathrm{met}(M)\times C^\infty(M)\times\mathbb{R}_+$ *defined in* (1.33)*. Its evolution is given by*

$$\begin{aligned}\delta\mathcal{W}_\infty(v,h,\xi)=&\int_M\Big[(\xi g^{\alpha\beta}-\tau v^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f-\frac{1}{2\tau}g_{\alpha\beta})+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}\Big)\Big(R_g-|\nabla f|^2\\&+2\Delta_g f+\frac{f-n-1}{\tau}\Big)\Big]u\,\mathrm{dV}+\int_{\partial M}\Big[\xi\left(2H+e_0 f\right)-\tau\big(\mathcal{A}^{ij}v_{ij}+v^{00}e_0 f\big)\\&+2\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}\Big)e_0 f+2\tau H\Big(\frac{g^{ij}v_{ij}}{2}-h-\frac{n\xi}{2\tau}\Big)\Big]u\,\mathrm{dA}\,.\end{aligned}$$

Proof. Observe that the functional in (1.33) can be decomposed as

$$\mathcal{W}_\infty(g,f,\tau)=\frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}I_\infty(g,f)+\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int_M(f-n)e^{-f}\,\mathrm{dV}\,.$$

Moreover, we can calculate the variation $\delta\mathcal{W}_\infty$ at (g,f,τ) in the direction of (v,h,ξ) as follows

$$\delta\mathcal{W}_\infty(v,h,\xi)=\delta\mathcal{W}_\infty(0,0,\xi)+\delta\mathcal{W}_\infty(v,h,0).$$

So,

$$\delta\mathcal{W}_\infty(v,h,\xi)=\delta\Big(\frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\Big)(0,0,\xi)I_\infty+\delta\Big(\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\Big)(0,0,\xi)\int_M(f-n)e^{-f}\,\mathrm{dV}$$
$$+\frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\delta I_\infty(v,h)+\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\delta\Big(\int_M(f-n)e^{-f}\,\mathrm{dV}\Big)(v,h).$$

Now, we compute some terms of this equation. The first one of them is

$$\delta\Big(\frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\Big)(0,0,\xi)I_\infty=\big(1-\frac{n}{2}\big)\frac{\xi}{(4\pi\tau)^{\frac{n}{2}}}I_\infty$$
$$=\int_M\big(1-\frac{n}{2}\big)\xi(R_g+|\nabla f|^2)u\,\mathrm{dV}+2\int_{\partial M}\big(1-\frac{n}{2}\big)\xi Hu\,\mathrm{dA}\,.$$

The second one is (see Proposition 1.15)

$$\frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\delta I_\infty(v,h)=\int_M\Big[-\tau v^{\alpha\beta}(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f)+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)(R_g-|\nabla f|^2+2\Delta_g f)\Big]u\,\mathrm{dV}$$
$$+\int_{\partial M}\tau\Big[2\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)e_0f-\mathcal{A}^{ij}v_{ij}-v^{00}e_0f+2H\Big(\frac{g^{ij}v_{ij}}{2}-h\Big)\Big]u\,\mathrm{dA}\,.$$

The third one is

$$\delta\Big(\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\Big)(0,0,\xi)\int_M(f-n)e^{-f}\,\mathrm{dV}=-\frac{n\xi}{2\tau}\int_M(f-n)u\,\mathrm{dV}\,.$$

The fourth one is

$$\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\delta\Big(\int_M(f-n)e^{-f}\,\mathrm{dV}\Big)(v,h)=\int_M\Big(h+(f-n)\big(\frac{v^\alpha{}_\alpha}{2}-h\big)\Big)u\,\mathrm{dV}\,.$$

By a combination of all these terms, we get

$$\begin{aligned}
&\delta\mathcal{W}_\infty(v,h,\xi)\\
&=\int_M\Big[-\tau v^{\alpha\beta}(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f)+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)(R_g-|\nabla f|^2+2\Delta_g f)\\
&\quad-\frac{n\xi}{2\tau}(f-n)+\xi(R_g+|\nabla f|^2)-\frac{n}{2}\xi(R_g+|\nabla f|^2)+h+(f-n)\big(\frac{v^\alpha{}_\alpha}{2}-h\big)\Big]u\,\mathrm{dV}\\
&\quad+\int_{\partial M}\Big[2\xi H+2\tau\big(\frac{v^\alpha{}_\alpha}{2}-h\big)e_0f-\tau\mathcal{A}^{ij}v_{ij}-\tau v^{00}e_0f+2\tau H\Big(\frac{g^{ij}v_{ij}}{2}-h-\frac{n}{2\tau}\xi\Big)\Big]u\,\mathrm{dA}\,.
\end{aligned}$$

Now absorb $(f-n)\big(\frac{v^\alpha{}_\alpha}{2}-h\big)$ into the third bracket of the terms on the first line to get

$$\begin{aligned}
&\delta\mathcal{W}_\infty(v,h,\xi)\\
&=\int_M\Big[(-\tau v^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f)+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)\Big(R_g-|\nabla f|^2+2\Delta_g f+\frac{f-n}{\tau}\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{n\xi}{2\tau}(f-n)+\xi(R_g+|\nabla f|^2)-\frac{n}{2}\xi(R_g+|\nabla f|^2)+h\Big]u\,\mathrm{dV}\\
&+\int_{\partial M}\Big[2\xi H+2\tau(\frac{v^\alpha{}_\alpha}{2}-h)e_0f-\tau\mathcal{A}^{ij}v_{ij}-\tau v^{00}e_0f+2\tau H\Big(\frac{g^{ij}v_{ij}}{2}-h-\frac{n}{2\tau}\xi\Big)\Big]u\,\mathrm{dA}\,.
\end{aligned}
$$

Now, use $\xi g^{\alpha\beta}$ into the first bracket of the terms on the first line to get

$$
\begin{aligned}
&\delta\mathcal{W}_\infty(v,h,\xi)\\
&=\int_M\Big[(-\tau v^{\alpha\beta}+\xi g^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f)+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)\Big(R_g-|\nabla f|^2+2\Delta_g f+\frac{f-n}{\tau}\Big)\\
&-\frac{n\xi}{2\tau}(f-n)+\xi(-\Delta_g f+|\nabla f|^2)-\frac{n}{2}\xi(R_g+|\nabla f|^2)+h\Big]u\,\mathrm{dV}\\
&+\int_{\partial M}\Big[2\xi H+2\tau(\frac{v^\alpha{}_\alpha}{2}-h)e_0f-\tau\mathcal{A}^{ij}v_{ij}-\tau v^{00}e_0f+2\tau H\Big(\frac{g^{ij}v_{ij}}{2}-h-\frac{n}{2\tau}\xi\Big)\Big]u\,\mathrm{dA}\,.
\end{aligned}
$$

Regrouping the terms on the first and second lines, we obtain

$$
\begin{aligned}
&\delta\mathcal{W}_\infty(v,h,\xi)\\
&=\int_M\Big[(-\tau v^{\alpha\beta}+\xi g^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f)+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}\Big)\Big(R_g-|\nabla f|^2+2\Delta_g f+\frac{f-n}{\tau}\Big)\\
&+\xi(-\Delta_g f+|\nabla f|^2)-n\xi|\nabla f|^2+n\xi\Delta_g f+h\Big]u\,\mathrm{dV}\\
&+\int_{\partial M}\Big[2\xi H+2\tau(\frac{v^\alpha{}_\alpha}{2}-h)e_0f-\tau\mathcal{A}^{ij}v_{ij}-\tau v^{00}e_0f+2\tau H\Big(\frac{g^{ij}v_{ij}}{2}-h-\frac{n}{2\tau}\xi\Big)\Big]u\,\mathrm{dA}\,.
\end{aligned}
$$

Using $-\frac{1}{2\tau}g_{\alpha\beta}$ into the second bracket of the terms on the first line together with the fact that

$$
\frac{1}{2\tau}g_{\alpha\beta}(-\tau v^{\alpha\beta}+\xi g^{\alpha\beta})=-\frac{1}{2}v^\alpha{}_\alpha+\frac{n}{2\tau}\xi
$$

and simplifying, one has

$$
\begin{aligned}
&\delta\mathcal{W}_\infty(v,h,\xi)\\
&=\int_M\Big[(-\tau v^{\alpha\beta}+\xi g^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f-\frac{1}{2\tau}g_{\alpha\beta})+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}\Big)\Big(R_g-|\nabla f|^2+2\Delta_g f\\
&+\frac{f-n-1}{\tau}\Big)+(n-1)\xi(\Delta_g f-|\nabla f|^2)\Big]u\,\mathrm{dV}\\
&+\int_{\partial M}\Big[2\xi H+2\tau(\frac{v^\alpha{}_\alpha}{2}-h)e_0f-\tau\mathcal{A}^{ij}v_{ij}-\tau v^{00}e_0f+2\tau H\Big(\frac{g^{ij}v_{ij}}{2}-h-\frac{n}{2\tau}\xi\Big)\Big]u\,\mathrm{dA}\,.
\end{aligned}
$$

Since $\int_M(|\nabla f|^2-\Delta_g f)e^{-f}\,\mathrm{dV}=\int_M\Delta_g e^{-f}\,\mathrm{dV}=\int_{\partial M}e_0fe^{-f}\,\mathrm{dA}$ we get

$$
\begin{aligned}
&\delta\mathcal{W}_\infty(v,h,\xi)\\
&=\int_M\Big[(-\tau v^{\alpha\beta}+\xi g^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f-\frac{1}{2\tau}g_{\alpha\beta})+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}\Big)\Big(R_g-|\nabla f|^2+2\Delta_g f
\end{aligned}
$$

$$+\frac{f-n-1}{\tau}\Big)\Big]u\,\mathrm{dV}+\int_{\partial M}\Big[2\xi H-(n-1)\xi e_0 f+2\tau\big(\frac{v^\alpha{}_\alpha}{2}-h\big)e_0 f-\tau\mathcal{A}^{ij}v_{ij}-\tau v^{00}e_0 f$$
$$+2\tau H\Big(\frac{g^{ij}v_{ij}}{2}-h-\frac{n}{2\tau}\xi\Big)\Big]u\,\mathrm{dA}\,.$$

Simplifying the terms on the boundary integrand we get the result of the proposition. $\square$

We observe that $\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}$ vanishes identically on M if and only if the measure $dm=u\,\mathrm{dV}$ remains fixed on M, since $\delta(u\,\mathrm{dV})=(\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau})u\,\mathrm{dV}$. In particular, as $v^\alpha{}_\alpha=g^{ij}v_{ij}+v^{00}$ we get

$$\frac{g^{ij}v_{ij}}{2}-h-\frac{n\xi}{2\tau}=\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}-\frac{1}{2}v^{00}=-\frac{1}{2}v^{00}. \tag{1.34}$$

Corollary 1.21. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and consider the $\mathcal{W}_\infty$*-type entropy *on* $\mathrm{met}(M)\times C^\infty(M)\times\mathbb{R}_+$ *defined in* (1.33)*. If $\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}=0$ on M, then*

$$\delta\mathcal{W}_\infty(v,h,\xi)=\int_M(\xi g^{\alpha\beta}-\tau v^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f-\frac{1}{2\tau}g_{\alpha\beta})u\,\mathrm{dV}+\tau\int_{\partial M}\Big(\frac{\xi}{\tau}(2H+e_0 f)-v^{ij}\mathcal{A}_{ij}$$
$$-v^{00}(H+e_0 f)\Big)u\,\mathrm{dA}\,.$$

Corollary 1.22 ([Per02, Sect. 3])**.** *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold without boundary, and consider the $\mathcal{W}_\infty$*-type entropy *on* $\mathrm{met}(M)\times C^\infty(M)\times\mathbb{R}_+$ *defined as in* (1.33)*. If $\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}=0$ on M, then*

$$\delta\mathcal{W}_\infty(v,h,\xi)=\int_M(\xi g^{\alpha\beta}-\tau v^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f-\frac{1}{2\tau}g_{\alpha\beta})u\,\mathrm{dV}\,.$$

Chapter 2

Mean curvature flow in an extended Ricci flow background

Let M be an $n(\geqslant 3)$-dimensional smooth manifold and let $(g(t), w(t))$ be a solution to the *extended Ricci flow*

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + 2\alpha_n \,\mathrm{d}w(t) \otimes \mathrm{d}w(t), \\ \frac{\partial}{\partial t} w(t) = \Delta_{g(t)} w(t), \end{cases} \tag{2.1}$$

in $M \times [0, T)$, for some initial value (g, w). Here $\alpha_n = (n-1)/(n-2)$ and $\mathrm{d}w(t) \otimes \mathrm{d}w(t)$ denotes the tensor product of the 1-form $\mathrm{d}w(t)$ by itself, which is metrically dual to gradient vector field $\nabla w(t)$ computed on $g(t)$ of a scalar smooth function $w(t)$ on M. For an account of extended Ricci flows, including proof of short-time existence of solutions to (2.1) on complete manifolds, we refer to List [Lis08, Thm. 4.1]. In this paper, List also showed that Hamilton's Ricci flow and the static Einstein vacuum equations are closely connected by extended Ricci flow, which justifies the value of the constant α_n. So, he provided an interesting and useful link from problems in low-dimensional topology and geometry to physical questions in general relativity (see [Lis08, p. 1010-1013] for details).

Associated to (2.1), List defined the $\mathcal{F}$*-type* functional on $\mathscr{P}(M) := \mathrm{met}(M) \times C^\infty(M) \times C^\infty(M)$ by

$$I_\infty^{\alpha_n}(g, f, w) = \int_M (R_\infty - \alpha_n |\nabla w|^2) e^{-f} \,\mathrm{dV}\,. \tag{2.2}$$

As mentioned before, Lott approached mean curvature flow in a Ricci flow background by introducing an analogue of Perelman's $\mathcal{F}$-functional for a manifold M with boundary ∂M, and then he obtained a weighted version I_∞ of the Gibbons-Hawking-York action.

In this chapter, we work in the setting of List by means of Lott's approach. The first step is to introduce the proper extension of (2.2) for manifolds with boundary.

Suppose M is an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M. We define the *weighted extended GHY-action* on $\mathscr{P}(M)$ given by

$$I_\infty^{\alpha_n}(g,f,w) = \int_M (R_\infty - \alpha_n|\nabla w|^2)e^{-f}\,\mathrm{dV} + 2\int_{\partial M} H_\infty e^{-f}\,\mathrm{dA}\,. \tag{2.3}$$

Our background is closely related to special solutions to (2.1), which we will study in detail now.

2.1 Gradient solitons to the extended Ricci flow

Special solutions to the extended Ricci flow come from gradient solitons. We describe such solutions on a background geometry, we follow as in Ph.D. thesis by List [Lis06, Sect. 2.2] in the line of Lott and Kleiner [KL08, Appx. C].

A *gradient soliton* to the extended Ricci flow is a self-similar solution $(\overline{g}(t), \overline{w}(t))$ of (2.1) given by

$$\begin{cases} \overline{g}(t) = \sigma(t)\psi_t^* g, & (2.4a)\\ \overline{w}(t) = \psi_t^* w, & (2.4b)\end{cases}$$

for some initial value (g,w), where ψ_t is a smooth one-parameter family of diffeomorphisms of M generated from the flow of $\nabla_g f/\sigma(t)$ computed on g, for some $f \in C^\infty(M)$, and σ is a smooth positive function on t. Gradient solitons to the extended Ricci flow are obtained as follows.

Proposition 2.1. *Let M be an $n(\geqslant 3)$-dimensional smooth manifold. Suppose there exists a triple (g,f,w) satisfying*

$$\begin{cases} \mathrm{Ric}_g + \mathrm{Hess}_g f - \alpha_n\,\mathrm{d}w\otimes\mathrm{d}w = \lambda g, & (2.5a)\\ \Delta_g w = \langle \nabla_g f, \nabla_g w\rangle_g & (2.5b)\end{cases}$$

for some $\lambda \in \mathbb{R}$ and $\alpha_n = (n-1)/(n-2)$. Take ψ_t the one-parameter family of diffeomorphisms generated by $Y_t = \frac{\nabla_g f}{\sigma(t)}$, with $\psi_0 = \mathrm{Id}$ and $\sigma(t) = 1-2\lambda t > 0$, where $t \in (-\infty, \frac{1}{2\lambda})$, for $\lambda > 0$; $t \in \mathbb{R}$, for $\lambda = 0$; and $t \in (\frac{1}{2\lambda}, +\infty)$, for $\lambda < 0$. Then, $(\overline{g}(t), \overline{w}(t)) = (\sigma(t)\psi_t^* g, \psi_t^* w)$ is a gradient soliton to the extended Ricci flow on M.

Proof. Setting $\overline{g}(t) = \sigma(t)\psi_t^* g$ and $\overline{w}(t) = \psi_t^* w$, one has

$$\begin{aligned}\frac{\partial}{\partial t}\overline{g}(t) &= \sigma'(t)\psi_t^* g + \sigma(t)\psi_t^*(\mathcal{L}_{Y_t} g) = \psi_t^*(-2\lambda g + 2\,\mathrm{Hess}_g f) = -2\psi_t^*(\lambda g - \mathrm{Hess}_g f)\\ &= -2\psi_t^*(\mathrm{Ric}_g - \alpha_n\,\mathrm{d}w\otimes\mathrm{d}w) = -2\,\mathrm{Ric}_{\overline{g}(t)} + 2\alpha_n\,\mathrm{d}\overline{w}(t)\otimes\mathrm{d}\overline{w}(t)\end{aligned}$$

and

$$\frac{\partial}{\partial t}\overline{w}(t) = \psi_t^*(\mathcal{L}_{Y_t} w) = \frac{1}{\sigma(t)}\psi_t^*\mathcal{L}_{\nabla_g f} w = \frac{1}{\sigma(t)}\psi_t^*\langle \nabla_g f, \nabla_g w\rangle_g = \frac{1}{\sigma(t)}\psi_t^*\Delta_g w = \Delta_{\overline{g}(t)}\overline{w}(t).$$

This completes the proof. $\square$

We observe that a gradient soliton to the extended Ricci flow on M can be characterized by means Proposition 2.1. Indeed, if $(\overline{g}(t), \overline{w}(t))$ is given by (2.4a) (as well (2.4b)) and satisfies (2.1), then by a straightforward computation it satisfies (2.5a) (as well (2.5b)). Thus, we can assume that a gradient soliton on M is a triple (g, f, w) as in Proposition 2.1. It is *steady* if $\lambda = 0$, *shrinking* if $\lambda > 0$ and *expanding* if $\lambda < 0$. The function f is called the potential function.

Now, by setting $\overline{f}(t) = \psi_t^* f$, using (2.5a), (2.5b) and conformal theory, we obtain

$$\begin{cases} \mathrm{Ric}_{\overline{g}} + \mathrm{Hess}_{\overline{g}} \overline{f} - \alpha_n \, d\overline{w} \otimes d\overline{w} = \frac{\lambda}{\sigma(t)} \overline{g}, \\ \Delta_{\overline{g}} \overline{w} = \langle \nabla_{\overline{g}} \overline{f}, \nabla_{\overline{g}} \overline{w} \rangle_{\overline{g}}. \end{cases}$$

Moreover, by scaling $\overline{g}$ one can normalize $\lambda = 1/2$ in the shrinking case and, $\lambda = -1/2$ in the expanding case. For $\lambda = 1/2$, $\sigma(t) = 1 - t > 0$ implies $t < 1$. Setting $s = t - 1$, we have $s + 1 = t < 1$, i.e., $s < 0$, and then, $\overline{g}(s) = \sigma(s) \psi_s^* g$, with $\sigma(s) = -s$ and $\psi_{-1} = \mathrm{Id}$. For $\lambda = -1/2$, $\sigma(t) = 1 + t > 0$ implies $t > -1$. Setting $s = t + 1$, we have $s - 1 = t > -1$, i.e., $s > 0$, and then, $\overline{g}(s) = \sigma(s) \psi_s^* g$, with $\sigma(s) = s$ and $\psi_1 = \mathrm{Id}$. So, we immediately obtain the next proposition.

Proposition 2.2. *Consider an $n(\geqslant 3)$-dimensional smooth manifold M, and let $(\overline{g}(t), \overline{w}(t))$ be a gradient soliton to the extended Ricci flow in M. The following identities hold for all time t:*

$$\begin{cases} \mathrm{Ric}_{\overline{g}} + \mathrm{Hess}_{\overline{g}} \overline{f} - \alpha_n \, d\overline{w} \otimes d\overline{w} = \dfrac{c}{2t} \overline{g}, & (2.6a) \\ \Delta_{\overline{g}} \overline{w} = \langle \nabla_{\overline{g}} \overline{f}, \nabla_{\overline{g}} \overline{w} \rangle_{\overline{g}} & (2.6b) \end{cases}$$

where $c = 0$ in the steady case (for $t \in \mathbb{R}$ and $\psi_0 = \mathrm{Id}$), $c = -1$ in the shrinking case (for $t \in (-\infty, 0)$ and $\psi_{-1} = \mathrm{Id}$) and $c = -1$ in the expanding case (for $t \in (0, +\infty)$ and $\psi_1 = \mathrm{Id}$), besides

$$\frac{\partial}{\partial t} \overline{f} = \|\nabla_{\overline{g}} \overline{f}\|_{\overline{g}}^2. \tag{2.7}$$

The function $\overline{f}$ is still called the potential function.

2.2 Evolution of weighted extended GHY-action

In this subsection, we obtain a variational formula for the weighted extended GHY-action $I_\infty^{\alpha_n}$ on $\mathscr{P}(M)$, where M is an $n(\geqslant 3)$-dimensional smooth manifold with boundary ∂M. In the next subsection, we apply this formula to derive the evolution equations of $I_\infty^{\alpha_n}$ under the Perelman's modified extended Ricci flow in $M \times [0, T)$.

Recall the following notations $\delta g = v$ and $\delta f = h$, and denote $\delta w = \vartheta$, where $\delta := \frac{\partial}{\partial t}\big|_{t=0}$.

The *variation* of $I_\infty^{\alpha_n}$ in direction $(v,h,\vartheta)\in S^2(M)\times C^\infty(M)\times C^\infty(M)$, is defined to be

$$\delta I_\infty^{\alpha_n}(v,h,\vartheta)(g,f,w)=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}I_\infty^{\alpha_n}(g+tv,f+th,w+t\vartheta).$$

Moreover, recall that $\frac{v^\alpha{}_\alpha}{2}-h$ vanishes identically on M if and only if the measure $\mathrm{d}m=e^{-f}\,\mathrm{dV}$ remains fixed on M, since $\delta(e^{-f}\,\mathrm{dV})=(\frac{v^\alpha{}_\alpha}{2}-h)e^{-f}\,\mathrm{dV}$.

Proposition 2.3. *Let $I_\infty^{\alpha_n}$ be the weighted extended GHY-action on $\mathscr{P}(M)$ defined in (2.3). Then, the following equality holds:*

$$\begin{aligned}\delta I_\infty^{\alpha_n}(v,h,\vartheta)=&\int_M\Big[v^{\alpha\beta}(-R_{\alpha\beta}-\nabla_\alpha\nabla_\beta f+\alpha_n\nabla_\alpha w\nabla_\beta w)+\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)(R_g-|\nabla f|^2+2\Delta_g f\\&-\alpha_n|\nabla w|^2)+2\alpha_n\vartheta(\Delta_g w-\langle\nabla w,\nabla f\rangle)\Big]e^{-f}\,\mathrm{dV}+\int_{\partial M}\Big[2\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)e_0f-v^{ij}\mathcal{A}_{ij}\\&-v^{00}e_0f+2\alpha_n\vartheta e_0w+2H\Big(\frac{g^{ij}v_{ij}}{2}-h\Big)\Big]e^{-f}\,\mathrm{dA}\,.\end{aligned}$$

In addition, if $\frac{v^\alpha{}_\alpha}{2}-h=0$ on M, then

$$\begin{aligned}\delta I_\infty^{\alpha_n}(v,h,\vartheta)=&\int_M v^{\alpha\beta}(-R_{\alpha\beta}-\nabla_\alpha\nabla_\beta f+\alpha_n\nabla_\alpha w\nabla_\beta w)+2\alpha_n\vartheta\big(\Delta_g w-\langle\nabla w,\nabla f\rangle\big)e^{-f}\,\mathrm{dV}\\&+\int_{\partial M}\Big(2\alpha_n\vartheta e_0w-v^{ij}\mathcal{A}_{ij}-v^{00}(H+e_0f)\Big)e^{-f}\,\mathrm{dA}\,.\end{aligned}$$

Proof. Notice that

$$I_\infty^{\alpha_n}(g,f,w)=\int_M\Big(R_\infty-\alpha_n|\nabla w|_g^2\Big)e^{-f}\,\mathrm{dV}+2\int_{\partial M}H_\infty e^{-f}\,\mathrm{dA}=I_\infty(g,f)-\alpha_nI_1(g,f,w),$$

where $I_1(g,f,w):=\int_M|\nabla w|_g^2e^{-f}\,\mathrm{dV}$. Hence, it is enough to compute $\delta I_1(v,h,\vartheta)$ since we know $\delta I_\infty(v,h)$ from Proposition 1.15. First, we have

$$\delta I_1=\int_M\Big[\delta\big(|\nabla w|^2\big)+|\nabla w|^2\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)\Big]e^{-f}\,\mathrm{dV}\,.$$

A straightforward computation gives us

$$\delta\big(|\nabla w|^2\big)=-g^{\alpha\gamma}v_{\gamma\zeta}g^{\beta\zeta}\nabla_\alpha w\nabla_\beta w+2g^{\alpha\beta}\nabla_\alpha\vartheta\nabla_\beta w.$$

So,

$$\delta I_1=\int_M\Big[-v^{\alpha\beta}\nabla_\alpha w\nabla_\beta we^{-f}+2g^{\alpha\beta}\nabla_\alpha\vartheta\nabla_\beta we^{-f}+|\nabla w|^2\Big(\frac{v^\alpha{}_\alpha}{2}-h\Big)e^{-f}\Big]\,\mathrm{dV}\,.$$

Integration by parts implies

$$\delta I_1 = \int_M \Big[-v^{\alpha\beta}\nabla_\alpha w \nabla_\beta w e^{-f} + 2g^{\alpha\beta}\nabla_\alpha\big(\vartheta\nabla_\beta w e^{-f}\big) - 2g^{\alpha\beta}\vartheta\nabla_\alpha\nabla_\beta w e^{-f} \\ - 2g^{\alpha\beta}\vartheta\nabla_\gamma w\Gamma^\gamma_{\alpha\beta}e^{-f} + 2g^{\alpha\beta}\vartheta\nabla_\beta w\nabla_\alpha f e^{-f} + |\nabla w|^2\Big(\frac{v^\alpha_\alpha}{2} - h\Big)e^{-f}\Big]\,\mathrm{dV}\,.$$

Since

$$\operatorname{div}(\vartheta e^{-f}\nabla w) = g^{\alpha\beta}\nabla_\alpha\big(\vartheta\nabla_\beta w e^{-f}\big) - g^{\alpha\beta}\vartheta\nabla_\gamma w\Gamma^\gamma_{\alpha\beta}e^{-f},$$

by Stokes' theorem, we get

$$\delta I_1 = \int_M \Big[-v^{\alpha\beta}\nabla_\alpha w\nabla_\beta w - 2\vartheta g^{\alpha\beta}\nabla_\alpha\nabla_\beta w e^{-f} + 2\vartheta g^{\alpha\beta}\nabla_\beta w\nabla_\alpha f + |\nabla w|^2\Big(\frac{v^\alpha_\alpha}{2} - h\Big)\Big] e^{-f}\,\mathrm{dV} \\ - 2\int_{\partial M}\vartheta e_0 w\, e^{-f}\,\mathrm{dA}\,.$$

The proposition is then a consequence of the previous equation and Proposition 1.15. □

Remark 2.4. If M has no boundary, then Proposition 2.3 appears in [Lis08, Sect. 3] as follows

$$\delta I^{\alpha_n}_\infty(v,h,\vartheta) = \int_M \Big[v^{\alpha\beta}(-R_{\alpha\beta} - \nabla_\alpha\nabla_\beta f + \alpha_n\nabla_\alpha w\nabla_\beta w) + \Big(\frac{v^\alpha_\alpha}{2} - h\Big)(R_g - |\nabla f|^2 \\ + 2\Delta_g f - \alpha_n|\nabla w|^2) + 2\alpha_n\vartheta(\Delta_g w - \langle\nabla w,\nabla f\rangle)\Big]e^{-f}\,\mathrm{dV}\,.$$

In addition, if $\frac{v^\alpha_\alpha}{2} - h = 0$ on M, then

$$\delta I^{\alpha_n}_\infty(v,h,\vartheta) = \int_M \Big[v^{\alpha\beta}(-R_{\alpha\beta} - \nabla_\alpha\nabla_\beta f + \alpha_n\nabla_\alpha w\nabla_\beta w) + 2\alpha_n\vartheta(\Delta_g w - \langle\nabla w,\nabla f\rangle)\Big]e^{-f}\,\mathrm{dV}\,.$$

In the same line of Remark 1.12, List showed that

$$\begin{cases} \frac{\partial}{\partial t}g(t) = -2(\mathrm{Ric}_{g(t)} + \mathrm{Hess}_{g(t)}f(t) - \alpha_n\,\mathrm{d}w(t)\otimes\mathrm{d}w(t)), \\ \frac{\partial}{\partial t}w(t) = \Delta_{g(t)}w(t) - \langle\nabla w(t),\nabla f(t)\rangle_{g(t)}, \\ \frac{\partial}{\partial t}f(t) = -\Delta_{g(t)}f(t) - R_{g(t)} + \alpha_n|\nabla w(t)|^2_{g(t)}, \end{cases} \tag{2.8}$$

has a solution in $M\times[a,b]$ (see List [Lis06, Sect. 2.1] for details), so that one can also think that $I^{\alpha_n}_\infty$ has a gradient-like structure on

$$\mathscr{C}_1 := \Big\{(v,\vartheta)\in S^2(M)\times C^\infty(M) : v = \delta g\ ,\ \vartheta = \delta w \text{ and } \delta(e^{-f}\,\mathrm{dV}) = 0\Big\}.$$

We call system (2.8) a *Perelman's modified extended Ricci flow* in $M\times[a,b]$.

The next corollary provides boundary conditions to couple in (2.8).

Corollary 2.5. *Let $I_\infty^{\alpha_n}$ be the weighted extended GHY-action on $\mathscr{P}(M)$ defined in (2.3). If the induced metric on ∂M is fixed, then the critical points of $I_\infty^{\alpha_n}$ constraint to $\mathscr{C}_1$ are gradient steady solitons on M that satisfy $H+e_0 f=0$ and $e_0 w=0$ on ∂M.*

Proof. First, note that

$$\langle v, e_0^\flat \otimes e_0^\flat\rangle = v^{\alpha\beta}(e_0^\flat \otimes e_0^\flat)_{\alpha\beta} = v^{\alpha\beta}\langle e_0,\partial_\alpha\rangle\langle e_0,\partial_\beta\rangle = v^{00}.$$

By hypotheses we have $\frac{v^\alpha{}_\alpha}{2}-h=0$ on M and $v_{ij}=0$ on ∂M which allows us to use Proposition 2.3 to obtain

$$\begin{aligned}\int_M \Big(\big\langle v, -\mathrm{Ric}_g - \mathrm{Hess}_g f + \alpha_n \,\mathrm{d}w\otimes \mathrm{d}w\big\rangle + 2\alpha_n \vartheta\left(\Delta_g w - \langle \nabla w, \nabla f\rangle\right)\Big) e^{-f}\,\mathrm{dV} \\ + \int_{\partial M}\Big(2\alpha_n\vartheta e_0 w - \big\langle v, (H+e_0 f) e_0^\flat\otimes e_0^\flat\big\rangle\Big) e^{-f}\,\mathrm{dA} = 0,\end{aligned} \tag{2.9}$$

for all $(v,\vartheta)\in\mathscr{C}_1$. We first start assuming $(v,\vartheta)\in\mathscr{C}_{1c}$. Then

$$\int_M \Big(\big\langle v, \alpha_n \,\mathrm{d}w\otimes \mathrm{d}w - \mathrm{Ric}_g - \mathrm{Hess}_g f\big\rangle + 2\alpha_n \vartheta\left(\Delta_g w - \langle \nabla w, \nabla f\rangle\right)\Big) e^{-f}\,\mathrm{dV} = 0.$$

Therefore (g,w,f) must be a gradient steady soliton. So, again by (2.9) we get

$$\int_{\partial M}\Big(2\alpha_n\vartheta e_0 w - \big\langle v, (H+e_0 f) e_0^\flat\otimes e_0^\flat\big\rangle\Big) e^{-f}\,\mathrm{dA} = 0,$$

for all $(v,\vartheta)\in\mathscr{C}_1$. Hence $H+e_0 f=0$ and $e_0 w=0$ on ∂M. □

Corollary 2.6. *Let $I_\infty^{\alpha_n}$ be the weighted extended GHY-action on $\mathscr{P}(M)$ defined in (2.3). The critical points of $I_\infty^{\alpha_n}$ constraint to $\mathscr{C}_1$ are gradient steady solitons on M with totally geodesic boundary satisfying the conditions $e_0 f=0$ and $e_0 w=0$ on ∂M.*

Proof. The argument is very similar to the proof of Corollary 2.5. Suppose that it has already been proven that (g,w,f) is a gradient steady soliton, then

$$\int_{\partial M}\Big(2\alpha_n\vartheta e_0 w - \big\langle v, \mathcal{A} + (H+e_0 f) e_0^\flat\otimes e_0^\flat\big\rangle\Big) e^{-f}\,\mathrm{dA} = 0, \tag{2.10}$$

for all $(v,\vartheta)\in\mathscr{C}_1$ from which we obtain that the critical points are gradient steady solitons on M with totally geodesic boundary satisfying the conditions $e_0 f=0$ and $e_0 w=0$ on ∂M. □

Remark 2.7. If w is constant, then we recover Remark 1.19.

2.2.1 Time-derivative of the weighted extended GHY-action under Perelman's modified extended Ricci flow

We begin with a general result from Gauss and Weingarten formulas. For it, assume that holds $H + e_0 f = 0$ on ∂M, then one has

$$\nabla_i \nabla_j f = \widehat{\nabla}_i \widehat{\nabla}_j f + H\mathcal{A}_{ij} \tag{2.11}$$

and

$$\nabla_i \nabla_0 f = -\widehat{\nabla}_i H + \mathcal{A}^k{}_i \widehat{\nabla}_k f. \tag{2.12}$$

Indeed, Gauss formula (see [DT19, p. 3]) implies

$$\nabla_i \nabla_j f = \partial_i \partial_j f - \left(\nabla_{\partial_i} \partial_j\right) f = \partial_i \partial_j f - \left(\widehat{\nabla}_{\partial_i} \partial_j + \mathcal{A}_{ij} e_0\right) f = \widehat{\nabla}_i \widehat{\nabla}_j f + H\mathcal{A}_{ij}.$$

Since $g(\nabla_i e_0, \nabla f) = g(\nabla_i e_0, \widehat{\nabla} f + e_0 f e_0) = -\nabla^k f \mathcal{A}_{ik}$, we get

$$\nabla_i \nabla_0 f = \partial_i \partial_0 f - \left(\nabla_{\partial_i} e_0\right) f = -\widehat{\nabla}_i H + \mathcal{A}^k{}_i \widehat{\nabla}_k f.$$

This finishes our claim.

Next, we compute the time-derivative of $I_\infty^{\alpha_n}$ under Perelman's modified extended Ricci flow.

Proposition 2.8. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and let $I_\infty^{\alpha_n}$ be the weighted extended GHY-action on $\mathscr{P}(M)$ defined in (2.3). If $(g(t), w(t)) \in \mathrm{met}(M) \times C^\infty(M)$, $t \in [0,T)$ evolves by*

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2(\mathrm{Ric} + \mathrm{Hess}\, f - \alpha_n \, \mathrm{d}w \otimes \mathrm{d}w), & (2.13a) \\ \dfrac{\partial}{\partial t} w = \Delta_g w - \langle \nabla f, \nabla w \rangle, & (2.13b) \end{cases}$$

and

$$\frac{\partial}{\partial t} f = -\Delta_g f - R_g + \alpha_n |\nabla w|^2 \tag{2.14}$$

in $M \times [0,T)$ satisfying $H + e_0 f = 0$ and $e_0 w = 0$ on ∂M. Then the following equality holds:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} I_\infty^{\alpha_n} = {} & 2\int_M \left(|\mathrm{Ric} + \mathrm{Hess}\, f - \alpha_n \, \mathrm{d}w \otimes \mathrm{d}w|^2 + \alpha_n \left(\Delta_g w - \langle \nabla w, \nabla f \rangle\right)^2 \right) e^{-f} \, \mathrm{dV} \\ & + 2\int_{\partial M} \Big(\widehat{\Delta} H - 2\langle \widehat{\nabla} f, \widehat{\nabla} H \rangle + \mathcal{A}(\widehat{\nabla} f, \widehat{\nabla} f) + \mathcal{A}^{ij}\mathcal{A}_{ij} H + \mathcal{A}^{ij} R_{ij} + 2R^{0i} \widehat{\nabla}_i f - \widehat{\nabla}_i R^{0i} \\ & - \alpha_n \mathcal{A}(\widehat{\nabla} w, \widehat{\nabla} w) \Big) e^{-f} \, \mathrm{dA}. \end{aligned}$$

In particular, if both $R_{ij}+\nabla_i\nabla_j f-\alpha_n\nabla_i w\nabla_j w$ and $R_{i0}+\nabla_i\nabla_0 f$ vanish on ∂M, then

$$\widehat{\Delta}H-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)-\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)+\mathcal{A}^{ij}\mathcal{A}_{ij}H+\mathcal{A}^{ij}R_{ij}+2R^{0i}\widehat{\nabla}_i f-\widehat{\nabla}_i R^{0i}=0.$$

Proof. By (2.13a) and (2.13b) we have $v_{\alpha\beta}=2(\alpha_n\nabla_\alpha w\nabla_\beta w-R_{\alpha\beta}-\nabla_\alpha\nabla_\beta f)$ and $\vartheta=\Delta_g w-\langle\nabla w,\nabla f\rangle$, respectively. Tracing the previous equation and using (2.14), we obtain $\frac{v^\alpha{}_\alpha}{2}-h=0$ on M, which allows us to use Proposition 2.3 to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_\infty^{\alpha_n}=&2\int_M\Big(|\operatorname{Ric}+\operatorname{Hess}f-\alpha_n\,\mathrm{d}w\otimes\mathrm{d}w|^2+\alpha_n\big(\Delta_g w-\langle\nabla w,\nabla f\rangle\big)^2\Big)e^{-f}\,\mathrm{dV}\\&+2\int_{\partial M}\big(\mathcal{A}^{ij}(R_{ij}+\nabla_i\nabla_j f-\alpha_n\nabla_i w\nabla_j w)\big)e^{-f}\,\mathrm{dA}\,.\end{aligned}$$

We claim that if $e_0 w=0$ on ∂M, then

$$\begin{aligned}&\mathcal{A}^{ij}(R_{ij}+\nabla_i\nabla_j f-\alpha_n\nabla_i w\nabla_j w)e^{-f}-\widehat{\nabla}_i\Big((R^{i0}+\nabla^i\nabla^0 f)e^{-f}\Big)\\=&\Big(\widehat{\Delta}H-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+\mathcal{A}^{ij}\mathcal{A}_{ij}H+\mathcal{A}^{ij}R_{ij}+2R^{0i}\widehat{\nabla}_i f-\widehat{\nabla}_i R^{0i}-\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)\Big)e^{-f}.\end{aligned}\tag{2.15}$$

Indeed, as $\mathcal{A}^{ij}(\widehat{\nabla}_i\widehat{\nabla}_j f)e^{-f}=\widehat{\nabla}_i\Big(\mathcal{A}^{ij}(\widehat{\nabla}_j f)e^{-f}\Big)-e^{-f}\Big(\widehat{\nabla}_i\mathcal{A}^{ij}\Big)\widehat{\nabla}_j f+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)e^{-f}$, we have

$$\begin{aligned}&\mathcal{A}^{ij}\Big(R_{ij}+\nabla_i\nabla_j f-\alpha_n\nabla_i w\nabla_j w\Big)e^{-f}\\&=\Big(\mathcal{A}^{ij}R_{ij}+\mathcal{A}^{ij}\widehat{\nabla}_i\widehat{\nabla}_j f+H\mathcal{A}^{ij}\mathcal{A}_{ij}-\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)\Big)e^{-f}\\&=\Big(\mathcal{A}^{ij}R_{ij}-\Big(\widehat{\nabla}_i\mathcal{A}^{ij}\Big)\widehat{\nabla}_j f+H\mathcal{A}^{ij}\mathcal{A}_{ij}+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)-\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)\Big)e^{-f}\\&\quad+\widehat{\nabla}_i\Big(\mathcal{A}^{ij}(\widehat{\nabla}_j f)e^{-f}\Big).\end{aligned}$$

Adding $0=\Big(\widehat{\Delta}H-\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle\Big)e^{-f}-\widehat{\nabla}_i(e^{-f}\widehat{\nabla}^i H)$ together with (1.12) leads

$$\begin{aligned}&\mathcal{A}^{ij}\Big(R_{ij}+\nabla_i\nabla_j f-\alpha_n\nabla_i w\nabla_j w\Big)e^{-f}\\&=\Big(\widehat{\Delta}H-\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}^{ij}R_{ij}-\Big(\widehat{\nabla}_i\mathcal{A}^{ij}\Big)\widehat{\nabla}_j f+H\mathcal{A}^{ij}\mathcal{A}_{ij}+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)-\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)\Big)e^{-f}\\&\quad+\widehat{\nabla}_i\Big(\Big(\mathcal{A}^{ij}\widehat{\nabla}_j f-\widehat{\nabla}^i H\Big)e^{-f}\Big)\\&=\Big(\widehat{\Delta}H-\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}^{ij}R_{ij}+R_0{}^j\widehat{\nabla}_j f-\widehat{\nabla}^j H\widehat{\nabla}_j f+H\mathcal{A}^{ij}\mathcal{A}_{ij}+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)-\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)\Big)e^{-f}\\&\quad+\widehat{\nabla}_i\Big(\Big(\mathcal{A}^{ij}\widehat{\nabla}_j f-\widehat{\nabla}^i H\Big)e^{-f}\Big)\\&=\Big(\widehat{\Delta}H-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}^{ij}R_{ij}+R_0{}^j\widehat{\nabla}_j f+H\mathcal{A}^{ij}\mathcal{A}_{ij}+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)-\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)\Big)e^{-f}\\&\quad+\widehat{\nabla}_i\Big(\Big(\mathcal{A}^{ij}\widehat{\nabla}_j f-\widehat{\nabla}^i H\Big)e^{-f}\Big).\end{aligned}\tag{2.16}$$

By equation (2.12), we get

$$\widehat{\nabla}_i\Big(\big(\mathcal{A}^{ij}\widehat{\nabla}_j f-\widehat{\nabla}^i H\big)e^{-f}\Big)=\widehat{\nabla}_i\Big(\nabla^0\nabla^i f e^{-f}\Big)=\widehat{\nabla}_i\Big(\big(R^{0i}+\nabla^0\nabla^i f\big)e^{-f}\Big)-\widehat{\nabla}_i\Big(R^{0i}e^{-f}\Big)$$
$$=\widehat{\nabla}_i\Big(\big(R^{0i}+\nabla^0\nabla^i f\big)e^{-f}\Big)+\Big(-\widehat{\nabla}_i R^{0i}+R^{0i}\widehat{\nabla}_i f\Big)e^{-f}.$$

We substitute this into (2.16) to establish our claim. Hence, the first part of the proposition follows from divergence theorem. In particular, if both $\big(R_{ij}+\nabla_i\nabla_j f-\alpha_n\nabla_i w\nabla_j w\big)|_{\partial M}$ and $\big(R_{i0}+\nabla_i\nabla_0 f\big)|_{\partial M}$ vanish, then from Eq. (2.15) the boundary integrand vanishes. □

Corollary 2.9 ([Lot12, Thm. 2]). *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and let I_∞ be the weighted GHY-action on* $\mathrm{met}(M)\times C^\infty(M)$ *defined in* (1.27). *If $(g(t))\in\mathrm{met}(M)$, $t\in[0,T)$ evolves by Perelman's modified Ricci flow in $M\times[0,T)$* (1.22) *satisfying $H+e_0 f=0$ on ∂M. Then the following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\infty=2\int_M|\mathrm{Ric}_g+\mathrm{Hess}_g f|^2e^{-f}\,\mathrm{dV}+2\int_{\partial M}(\widehat{\Delta}h-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+\mathcal{A}^{ij}\mathcal{A}_{ij}H+\mathcal{A}^{ij}R_{ij}$$
$$+2R^{0i}\widehat{\nabla}_i f-\widehat{\nabla}_i R^{0i})e^{-f}\,\mathrm{dA}\,.$$

2.2.2 Evolution equations for the boundary geometry under a Perelman's modified extended Ricci flow

In our next result, we establish the evolution equations of the geometric quantities of ∂M under Perelman's modified extended Ricci flow.

Proposition 2.10. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and let $I_\infty^{\alpha_n}$ be the weighted extended GHY-action on $\mathscr{P}(M)$ defined in* (2.3). *If $(g(t),w(t))\in$* $\mathrm{met}(M)\times C^\infty(M)$, $t\in[0,T)$ *evolves by*

$$\begin{cases}\dfrac{\partial}{\partial t}g=-2(\mathrm{Ric}+\mathrm{Hess}\,f-\alpha_n\,\mathrm{d}w\otimes\mathrm{d}w), & (2.17\mathrm{a})\\[2mm] \dfrac{\partial}{\partial t}w=\Delta_g w-\langle\nabla f,\nabla w\rangle, & (2.17\mathrm{b})\end{cases}$$

and

$$\frac{\partial}{\partial t}f=-\Delta_g f-R_g+\alpha_n|\nabla w|^2 \tag{2.18}$$

in $M\times[0,T)$ satisfying $H+e_0 f=0$ and $e_0 w=0$ on ∂M. Then, the following hold:

$$\frac{\partial}{\partial t}g_{ij}=-(\mathcal{L}_{\widehat{\nabla}f}g)_{ij}-2(R_{ij}-\alpha_n\widehat{\nabla}_i w\widehat{\nabla}_j w)-2H\mathcal{A}_{ij}, \tag{2.19}$$

$$\frac{\partial}{\partial t}w=\widehat{\Delta}w+\nabla_0\nabla_0 w-\mathcal{L}_{\widehat{\nabla}f}w, \tag{2.20}$$

$$\frac{\partial}{\partial t}\mathcal{A}_{ij} = (\widehat{\Delta}\mathcal{A})_{ij} - (\mathcal{L}_{\widehat{\nabla} f}\mathcal{A})_{ij} - \mathcal{A}^k{}_i R^l{}_{klj} - \mathcal{A}^k{}_j R^l{}_{kli} + 2\mathcal{A}^{kl}R_{kilj} - 2H\mathcal{A}_{ik}\mathcal{A}^k{}_j$$
$$+ \mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij} + \nabla_0 R_{0i0j} \tag{2.21}$$

and

$$\frac{\partial}{\partial t}H = \widehat{\Delta}H - \langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + 2\mathcal{A}^{ij}R_{ij} + \mathcal{A}^{ij}\mathcal{A}_{ij}H + \nabla_0 R_{00} - 2\alpha_n \mathcal{A}(\widehat{\nabla} w, \widehat{\nabla} w). \tag{2.22}$$

Proof. The proof is analogous to the one given for Theorem 3 in [Lot12]. Nevertheless, we shall present it here for the reader's convenience.

We start by substituting $\nabla_i\nabla_j f = \widehat{\nabla}_i\widehat{\nabla}_i f + H\mathcal{A}_{ij}$ (2.11) into equation (2.17a) to get

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= -2(R_{ij} + \widehat{\nabla}_i\widehat{\nabla}_j f + H\mathcal{A}_{ij} - \alpha_n\widehat{\nabla}_i w\widehat{\nabla}_j w)\\ &= -(\mathcal{L}_{\widehat{\nabla} f}g)_{ij} - 2(R_{ij} - \alpha_n\widehat{\nabla}_i w\widehat{\nabla}_j w) - 2H\mathcal{A}_{ij},\end{aligned}$$

which is equation (2.19). Likewise, equation (2.20) follows from Proposition 1.5.

To prove equation (2.21) we first observe that by (2.17a)

$$\frac{1}{2}v_{\alpha\beta} = -(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f - \alpha_n\nabla_\alpha w\nabla_\beta w).$$

It has already been calculated $\frac{\partial}{\partial t}\mathcal{A}_{ij}$ to arbitrary variation (see (1.28))

$$\delta\mathcal{A}_{ij} = \frac{1}{2}(\nabla_i v_{j0} + \nabla_j v_{i0} - \nabla_0 v_{ij}) + \frac{1}{2}v_{00}\mathcal{A}_{ij}.$$

Since $e_0 w = 0$ implies $v_{00} = R_{00} + \nabla_0\nabla_0 f$, the previous equation is rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{ij} = &-\nabla_i(R_{j0} + \nabla_j\nabla_0 f - \alpha_n\nabla_j w\nabla_0 w) - \nabla_j(R_{i0} + \nabla_i\nabla_0 f - \alpha_n\nabla_i w\nabla_0 w)\\ &+ \nabla_0(R_{ij} + \nabla_i\nabla_j f - \alpha_n\nabla_i w\nabla_j w) - \mathcal{A}_{ij}(R_{00} + \nabla_0\nabla_0 f).\end{aligned}$$

Now we will compute some terms of this equation. The first one of them is

$$\nabla_i\nabla_j\nabla_0 f = \widehat{\nabla}_i\nabla_j\nabla_0 f - \mathcal{A}_{ij}\nabla_0\nabla_0 f + \mathcal{A}^k{}_i\nabla_j\nabla_k f.$$

Replacing $\nabla_j\nabla_0 f = -\widehat{\nabla}_j H + \mathcal{A}^k{}_j\widehat{\nabla}_k f$ (see (2.12)), we obtain

$$\begin{aligned}\nabla_i\nabla_j\nabla_0 f &= \widehat{\nabla}_i\Big(-\widehat{\nabla}_j H + \mathcal{A}^k{}_j\widehat{\nabla}_k f\Big) - \mathcal{A}_{ij}\nabla_0\nabla_0 f + \mathcal{A}^k{}_i\Big(\widehat{\nabla}_j\widehat{\nabla}_k f + H\mathcal{A}_{jk}\Big)\\ &= \widehat{\nabla}_i\Big(-\widehat{\nabla}_j H + \mathcal{A}^k{}_j\widehat{\nabla}_k f\Big) - \mathcal{A}_{ij}\nabla_0\nabla_0 f + \mathcal{A}^k{}_i\widehat{\nabla}_j\widehat{\nabla}_k f + H\mathcal{A}^k{}_i\mathcal{A}_{jk}\\ &= -\widehat{\nabla}_i\widehat{\nabla}_j H + \Big(\widehat{\nabla}_i\mathcal{A}^k{}_j\Big)\widehat{\nabla}_k f + \mathcal{A}^k{}_j\widehat{\nabla}_i\widehat{\nabla}_k f - \mathcal{A}_{ij}\nabla_0\nabla_0 f + \mathcal{A}^k{}_i\widehat{\nabla}_j\widehat{\nabla}_k f + H\mathcal{A}^k{}_i\mathcal{A}_{jk}.\end{aligned}$$

The second one of them is due to Symmetry Lemma (see (1.1))

$$\nabla_0\nabla_i\nabla_j f-\nabla_j\nabla_i\nabla_0 f=\nabla_0\nabla_j\nabla_i f-\nabla_j\nabla_0\nabla_i f=-R_{0j\alpha i}\nabla^\alpha f=-R_{0jki}\widehat{\nabla}^k f-R_{0j0i}\nabla_0 f.$$

The third one of them is

$$\begin{aligned}\nabla_0\Big(\nabla_i w\nabla_j w\Big)&=\partial_0(\nabla_i w\nabla_j w)-\langle\nabla w,\nabla_{\partial_0}\partial_i\rangle\nabla_j w-\nabla_i w\langle\nabla w,\nabla_{\partial_0}\partial_j\rangle\\&=\Big(\nabla_0\nabla_i w+\langle\nabla w,\nabla_{\partial_0}\partial_i\rangle\Big)\nabla_j w+\nabla_i w\Big(\nabla_0\nabla_j w+\langle\nabla w,\nabla_{\partial_0}\partial_j\rangle\Big)\\&\quad-\langle\nabla w,\nabla_{\partial_0}\partial_i\rangle\nabla_j w-\nabla_i w\langle\nabla w,\nabla_{\partial_0}\partial_j\rangle\\&=\nabla_i\nabla_0 w\nabla_j w+\nabla_i w\nabla_j\nabla_0 w.\end{aligned}$$

By interchanging 0 and j we also obtain

$$\nabla_j\Big(\nabla_i w\nabla_0 w\Big)=\nabla_i\nabla_j w\nabla_0 w+\nabla_i w\nabla_0\nabla_j w.$$

All this implies that

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{ij}&=-\nabla_i R_{j0}-\nabla_j R_{i0}+\nabla_0 R_{ij}-\mathcal{A}_{ij}(R_{00}+\nabla_0\nabla_0 f)-\nabla_i\nabla_j\nabla_0 f-\nabla_j\nabla_i\nabla_0 f+\nabla_0\nabla_i\nabla_j f\\&\quad+\alpha_n\nabla_i(\nabla_j w\nabla_0 w)+\alpha_n\nabla_j(\nabla_i w\nabla_0 w)-\alpha_n\nabla_0(\nabla_i w\nabla_j w)\\&=-\nabla_i R_{j0}-\nabla_j R_{i0}+\nabla_0 R_{ij}-\mathcal{A}_{ij}(R_{00}+\nabla_0\nabla_0 f)+\widehat{\nabla}_i\widehat{\nabla}_j H-\Big(\widehat{\nabla}_i\mathcal{A}^k{}_j\Big)\widehat{\nabla}_k f\\&\quad-\mathcal{A}^k{}_j\widehat{\nabla}_i\widehat{\nabla}_k f+\mathcal{A}_{ij}\nabla_0\nabla_0 f-\mathcal{A}^k{}_i\widehat{\nabla}_j\widehat{\nabla}_k f-H\mathcal{A}^k{}_i\mathcal{A}_{jk}-R_{0jki}\widehat{\nabla}^k f+R_{0j0i}H\\&=\widehat{\nabla}_i\widehat{\nabla}_j H-(\widehat{\nabla}_i\mathcal{A}_{kj}-R_{0jik})\widehat{\nabla}^k f-\mathcal{A}^k{}_i\widehat{\nabla}_j\widehat{\nabla}_k f-\mathcal{A}^k{}_j\widehat{\nabla}_i\widehat{\nabla}_k f+R_{0i0j}H\\&\quad-\nabla_i R_{j0}-\nabla_j R_{i0}+\nabla_0 R_{ij}-\mathcal{A}_{ij}R_{00}-H\mathcal{A}^k{}_i\mathcal{A}_{jk}.\end{aligned}$$

Using Codazzi-Mainardi equation $R_{0jik}=\widehat{\nabla}_i\mathcal{A}_{jk}-\widehat{\nabla}_k\mathcal{A}_{ij}$ (see (1.11)) gives

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{ij}&=\widehat{\nabla}_i\widehat{\nabla}_j H-\Big(\widehat{\nabla}_k\mathcal{A}_{ij}\Big)\widehat{\nabla}^k f-\mathcal{A}^k{}_i\widehat{\nabla}_j\widehat{\nabla}_k f-\mathcal{A}^k{}_j\widehat{\nabla}_i\widehat{\nabla}_k f+R_{0i0j}H-\nabla_i R_{j0}-\nabla_j R_{i0}+\nabla_0 R_{ij}\\&\quad-\mathcal{A}_{ij}R_{00}-H\mathcal{A}^k{}_i\mathcal{A}_{jk}.\end{aligned}$$

Making $T=\mathcal{A}$ in (1.3) to obtain

$$\frac{\partial}{\partial t}\mathcal{A}_{ij}=\widehat{\nabla}_i\widehat{\nabla}_j H-\Big(\mathcal{L}_{\widehat{\nabla}f}\mathcal{A}\Big)_{ij}-\nabla_i R_{j0}-\nabla_j R_{i0}+\nabla_0 R_{ij}-\mathcal{A}_{ij}R_{00}+R_{0i0j}H-H\mathcal{A}^k{}_i\mathcal{A}_{jk}.$$

From Simons' identity (see (1.10)) we get

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{ij}&=(\widehat{\Delta}\mathcal{A})_{ij}-\Big(\mathcal{L}_{\widehat{\nabla}f}\mathcal{A}\Big)_{ij}-(\nabla_i R_{j0}-\widehat{\nabla}_i R_{j0})-(\nabla_j R_{i0}-\widehat{\nabla}_j R_{i0})-2\mathcal{A}_{ij}R_{00}+\mathcal{A}^k{}_i R_{0k0j}\\&\quad+\mathcal{A}^k{}_j R_{0k0i}+2\mathcal{A}^{kl}R_{kilj}-2H\mathcal{A}^k{}_i\mathcal{A}_{jk}+\mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij}+\nabla_0 R_{0i0j}.\end{aligned}$$

Since $\nabla_i R_{j0} = \widehat{\nabla}_i R_{j0} - \mathcal{A}_{ij} R_{00} + \mathcal{A}^k{}_i R_{jk}$ (see (1.2)), we conclude (2.21).

For finishing our proof, we will show equation (2.22). For it, note that

$$\delta H = -v_{ij}\mathcal{A}^{ij} + g^{ij}\delta\mathcal{A}_{ij}$$

and by (1.4) applied to $T = \mathcal{A}$ to obtain

$$g^{ij}(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A})_{ij} - 2\mathcal{A}^{ij}\widehat{\nabla}_i\widehat{\nabla}_j f = \widehat{\nabla}_{\widehat{\nabla} f}(g^{ij}\mathcal{A}_{ij}) = \langle \widehat{\nabla} f, \widehat{\nabla} H\rangle.$$

So,

$$\begin{aligned}
\frac{\partial}{\partial t}H &= 2(R_{ij} + \widehat{\nabla}_i\widehat{\nabla}_j f - \alpha_n \widehat{\nabla}_i w \widehat{\nabla}_j w + H\mathcal{A}_{ij})\mathcal{A}^{ij} + g^{ij}\Big(\frac{\partial}{\partial t}\mathcal{A}_{ij}\Big)\\
&= 2(R_{ij} + \widehat{\nabla}_i\widehat{\nabla}_j f + H\mathcal{A}_{ij})\mathcal{A}^{ij} + g^{ij}\Big((\widehat{\Delta}\mathcal{A})_{ij} - \Big(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A}\Big)_{ij} - \mathcal{A}^k{}_i R^l{}_{klj} - \mathcal{A}^k{}_j R^l{}_{kli}\\
&\quad + 2\mathcal{A}^{kl}R_{kilj} - 2H\mathcal{A}^k{}_i\mathcal{A}_{jk} + \mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij} + \nabla_0 R_{0i0j}\Big) - 2\alpha_n\mathcal{A}(\widehat{\nabla} w, \widehat{\nabla} w)\\
&= 2\mathcal{A}^{ij}R_{ij} + 2H\mathcal{A}^{ij}\mathcal{A}_{ij} + \widehat{\Delta} H - \Big(g^{ij}(\mathcal{L}_{\widehat{\nabla} f}\mathcal{A})_{ij} - 2\mathcal{A}^{ij}\widehat{\nabla}_i\widehat{\nabla}_j f\Big) - 2\mathcal{A}^{kj}\mathcal{A}_{jk}H + \mathcal{A}^{kl}\mathcal{A}_{kl}H\\
&\quad + \nabla_0 R_{00}\\
&= \widehat{\Delta} H - \langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + 2\mathcal{A}^{ij}R_{ij} + \mathcal{A}^{ij}\mathcal{A}_{ij}H + \nabla_0 R_{00} - 2\alpha_n\mathcal{A}(\widehat{\nabla} w, \widehat{\nabla} w).
\end{aligned}$$

This finishes the proof. □

As a consequence of Proposition 2.10, we have the following refinement of the formula obtained in Proposition 2.8.

Corollary 2.11. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and let $I_\infty^{\alpha_n}$ be the weighted extended GHY-action on $\mathscr{P}(M)$ defined in (2.3). If $(g(t), w(t)) \in \mathrm{met}(M) \times C^\infty(M)$, $t \in [0, T)$ evolves by*

$$\begin{cases} \frac{\partial}{\partial t} g = -2(\mathrm{Ric} + \mathrm{Hess}\, f - \alpha_n\, \mathrm{d}w \otimes \mathrm{d}w),\\ \frac{\partial}{\partial t} w = \Delta_g w - \langle \nabla f, \nabla w\rangle \end{cases} \tag{2.23}$$

and

$$\frac{\partial}{\partial t} f = -\Delta_g f - R_g + \alpha_n |\nabla w|^2 \tag{2.24}$$

in $M \times [0, T)$ satisfying $H + e_0 f$ and $e_0 w = 0$ on ∂M. Then the following identity holds:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I_\infty^{\alpha_n} &= 2\int_M \Big(\big|\mathrm{Ric} + \mathrm{Hess}\, f - \alpha_n\, \mathrm{d}w \otimes \mathrm{d}w\big|^2 + \alpha_n\big(\Delta_g w - \langle \nabla w, \nabla f\rangle\big)^2\Big) e^{-f}\, \mathrm{dV}\\
&\quad + 2\int_{\partial M}\Big(\frac{\partial}{\partial t}H - \langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f, \widehat{\nabla} f) + 2R^{0i}\widehat{\nabla}_i f - \frac{1}{2}\nabla_0 R - HR_{00} + \alpha_n\mathcal{A}(\widehat{\nabla} w, \widehat{\nabla} w)\Big) e^{-f}\mathrm{dA}.
\end{aligned}$$

In particular, if both $R_{ij}+\nabla_i\nabla_j f-\alpha_n\nabla_i w\nabla_j w$ and $R_{i0}+\nabla_i\nabla_0 f$ vanish on ∂M, then the integrand of ∂M vanishes.

Proof. From equation (2.22) of Proposition 2.10, the boundary integrand term of Proposition 2.8 can be rewritten as

$$\begin{aligned}
&\widehat{\Delta}H-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+\mathcal{A}^{ij}\mathcal{A}_{ij}H+\mathcal{A}^{ij}R_{ij}+2R^{0i}\widehat{\nabla}_i f-\widehat{\nabla}_i R^{0i}-\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)\\
&=\frac{\partial}{\partial t}H-\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+2R^{0i}\widehat{\nabla}_i f-\frac{1}{2}\nabla_0 R-HR_{00}+\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w).
\end{aligned}$$

The second contracted Bianchi identity and the fact that $\nabla_i R_{j0}=\widehat{\nabla}_i R_{j0}-\mathcal{A}_{ij}R_{00}+\mathcal{A}^k{}_i R_{jk}$ (see (1.2)) imply

$$\frac{1}{2}\nabla_0 R=\nabla_i R^{i0}+\nabla_0 R_{00}=\widehat{\nabla}_i R^{i0}+\mathcal{A}^{ij}R_{ij}-HR_{00}+\nabla_0 R_{00}.$$

The required integral formula follows from these two latter equations. If, in addition, both $R_{ij}+\nabla_i\nabla_j f-\alpha_n\nabla_i w\nabla_j w$ and $R_{i0}+\nabla_i\nabla_0 f$ vanish on ∂M, then by Proposition 2.8 the integrand of ∂M, namely

$$\frac{\partial}{\partial t}H-\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+2R^{0i}\widehat{\nabla}_i f-\frac{1}{2}\nabla_0 R-HR_{00}+\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w)$$

vanishes. □

2.3 Hypersurfaces in an extended Ricci flow background

In this section, we consider mean curvature flow in an extended Ricci flow background (see definition in Subsection 2.3.1). In particular, we recover the mean curvature flow in a Ricci flow background (or Ricci-mean curvature flow) by Lott. For explicit examples of Ricci-mean curvature flows, see Yamamoto [Yam18].

In Subsection 2.3.1, we translate to an evolving hypersurface Σ in an extended Ricci flow solution the results of the previous sections from a fixed manifold with boundary equipped with a Perelman's modified extended Ricci flow $(g(t),w(t))$.

In Subsection 2.3.2, we show that the differential Harnack-type expression vanishes on mean curvature solitons with additional assumption $e_0 w=0$ on Σ.

In Subsection 2.3.3, we study the mean curvature solitons more closely. For instance, in Theorem 2.23 we give a characterization of such solitons.

In Subsection 2.3.4, we give proof of the monotonicity of a Huisken-type functional under extended Ricci flow.

2.3.1 Mean curvature flow in an extended Ricci flow background

In this subsection, we shall consider mean curvature flows in the following context: let M be an $n(\geqslant 3)$-dimensional smooth manifold and let $(g(t), w(t))$ be an extended Ricci flow in $M \times [0, T)$. Given an $(n-1)$-dimensional smooth compact manifold Σ without boundary, and let $\{x(\cdot, t); t \in [0, T)\}$ be a smooth one-parameter family of immersions of Σ into M. For each $t \in [0, T)$, set $x_t = x(\cdot, t)$ and Σ_t for the hypersurface $x_t(\Sigma)$ of $(M, g(t))$, i.e.,

$$\Sigma_t := (x_t(\Sigma), g(t)),\ t \in [0, T),$$

and suppose that the family $\mathscr{F} := \{\Sigma_t\,; t \in [0, T)\}$ evolves under mean curvature flow, that is,

$$\begin{cases} \frac{\partial}{\partial t} x(p,t) = H(p,t) e(p,t), \\ x(p,0) = x_0(p), \end{cases}$$

where $H(p,t)$ and $e(p,t)$ are the mean curvature and the unit normal of Σ_t at the point $p \in \Sigma$, respectively. In this setting, we say that $\mathscr{F}$ is the *mean curvature flow in the* $(g(t), w(t))$-*extended Ricci flow background*. In the particular case $(g(t), w(t)) = (\overline{g}(t), \overline{w}(t))$ is a gradient soliton to the extended Ricci flow on M with potential function $\overline{f}$, a hypersurface $\Sigma_t \in \mathscr{F}$ is a *mean curvature soliton*, if

$$H(p,t) + e(p,t)\overline{f} = 0\ \forall p \in \Sigma_t.$$

Here, $e(\,\cdot\,, t)$ must be the inward unit normal vector field on Σ_t.

Remark 2.12. We highlight that the coupling of mean curvature flow with Ricci flow also was considered by Magni, Mantegazza and Tsatis (see [MMT13]). Moreover, Lott observed that mean curvature solitons for the mean curvature flow evolving in gradient Ricci soliton solutions arise quite naturally as f-minimal hypersurfaces.

Now we are working to prove the main results of this thesis.

Proposition 2.13. *Suppose that the family* $\mathscr{F} := \{\Sigma_t\,; t \in [0, T)\}$ *is a mean curvature flow in a* $(g(t), w(t))$*-extended Ricci flow background which satisfies* $e_0 w = 0$ *on* Σ_0*, where* e_0 *is the unit normal vector field on* Σ_0*. Then, the following evolution equations hold:*

$$\frac{\partial}{\partial t} g_{ij} = -2(R_{ij} - \alpha_n \widehat{\nabla}_i w \widehat{\nabla}_j w) - 2H\mathcal{A}_{ij}, \tag{2.25}$$

$$\frac{\partial}{\partial t} w = \widehat{\Delta} w + \nabla_0 \nabla_0 w, \tag{2.26}$$

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{A}_{ij} = {} & (\widehat{\Delta}\mathcal{A})_{ij} - \mathcal{A}^k{}_i R^l{}_{klj} - \mathcal{A}^k{}_j R^l{}_{kli} + 2\mathcal{A}^{kl} R_{kilj} - 2H\mathcal{A}_{ik}\mathcal{A}^k{}_j \\ & + \mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij} + \nabla_0 R_{0i0j} \end{aligned} \tag{2.27}$$

and

$$\frac{\partial}{\partial t}H = \widehat{\Delta}H + 2\mathcal{A}^{ij}R_{ij} + \mathcal{A}^{ij}\mathcal{A}_{ij}H + \nabla_0 R_{00} - 2\alpha_n \mathcal{A}(\widehat{\nabla}w, \widehat{\nabla}w). \tag{2.28}$$

Proof. In this proof, we follow [Lot12, Prop. 4] closely. First assume that $\Sigma_t = \partial X_t$ with each X_t compact. Given a time interval $[a,b]$, we can find a positive solution $u = e^{-f}$ on $\bigcup_{t\in[a,b]}(X_t \times \{t\}) \subset M \times [a,b]$ of the conjugate heat equation

$$\frac{\partial}{\partial t}u(t) = -\Delta_{g(t)}u(t) + R_{g(t)}u(t) - \alpha_n |\nabla w(t)|^2_{g(t)}u(t) \tag{2.29}$$

satisfying the boundary condition $e(\cdot,t)u = Hu$, by solving it backwards in time from $t = b$. (Choosing diffeomorphisms from $\{X_t\}$ to X_a, we can reduce the problem of solving (2.29) to a parabolic equation on a fixed domain with $e_0 w = 0$ on Σ_0).

Let $\{\phi_t\}_{t\in[a,b]}$ be the one-parameter family of diffeomorphisms generated by $\{-\nabla_{g(t)}f(t)\}$, with $\phi_a = \mathrm{Id}$. Then $\phi_t(X_a) = X_t$ for all t. By setting $\widetilde{g}(t) = \phi_t^* g(t)$, $\widetilde{w}(t) = \phi_t^* w(t)$ and $\widetilde{f}(t) = \phi_t^* f(t)$ we have that $\widetilde{g}(t)$, $\widetilde{w}(t)$ and $\widetilde{f}(t)$ are defined on X_a. We claim that

$$\begin{cases} \frac{\partial}{\partial t}\widetilde{g}(t) = -2(\mathrm{Ric}_{\widetilde{g}(t)} + \mathrm{Hess}_{\widetilde{g}(t)}\widetilde{f}(t) - \alpha_n \,\mathrm{d}\widetilde{w}(t)\otimes \mathrm{d}\widetilde{w}(t)), \\ \frac{\partial}{\partial t}\widetilde{w} = \Delta_{\widetilde{g}(t)}\widetilde{w}(t) - \langle \nabla\widetilde{w}(t), \nabla\widetilde{f}(t)\rangle_{\widetilde{g}(t)}, \end{cases}$$

and

$$\frac{\partial}{\partial t}\widetilde{f}(t) = -\Delta_{\widetilde{g}(t)}\widetilde{f}(t) - R_{\widetilde{g}(t)} + \alpha_n |\nabla\widetilde{w}(t)|^2_{\widetilde{g}(t)}$$

in $X_a \times [a,b]$ with $e_0 f + H = 0$ and $e_0 w = 0$ on $\partial X_a = \Sigma$. Indeed,

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{g} &= \phi_t^*\big(\frac{\partial}{\partial t}g\big) + \phi_t^*\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t} g = \phi_t^*\big(-2(\mathrm{Ric}_g - \alpha_n \,\mathrm{d}w\otimes \mathrm{d}w)\big) - \phi_t^*\mathcal{L}_{\left(\nabla_{g(t)}f(t)\right)} g \\ &= -2(\mathrm{Ric}_{\widetilde{g}} + \mathrm{Hess}_{\widetilde{g}}\widetilde{f} - \alpha_n \,\mathrm{d}\widetilde{w}\otimes \mathrm{d}\widetilde{w}), \end{aligned}$$

in X_a. For the second item, we have

$$\frac{\partial}{\partial t}\widetilde{w} = \phi_t^*\Big(\frac{\partial}{\partial t}w\Big) + \phi_t^*\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t} w = \phi_t^*\Big(\Delta w\Big) - \phi_t^*\mathcal{L}_{\left(\nabla_{g(t)}f(t)\right)} w = \Delta_{\widetilde{g}}\widetilde{w} - \langle \nabla\widetilde{w}, \nabla\widetilde{f}\rangle_{\widetilde{g}}.$$

Now, we use that $\Delta u = (|\nabla f|^2 - \Delta f)e^{-f}$ and (2.29) to obtain

$$\frac{\partial}{\partial t}\widetilde{f} = \phi_t^*\Big(\frac{\partial}{\partial t}f\Big) + \phi_t^*\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t} f = \phi_t^*\Big(|\nabla f|^2 - \Delta f - R + \alpha_n |\nabla w|^2\Big) - \phi_t^*\mathcal{L}_{\left(\nabla_{g(t)}f(t)\right)} f = -\Delta_{\widetilde{g}}\widetilde{f} - R_{\widetilde{g}} + \alpha_n |\nabla\widetilde{w}|^2_{\widetilde{g}}.$$

The boundary conditions follow from the fact that $e_0 u = Hu$ and $e_0 w = 0$ on Σ_0. Thus, $(\widetilde{g}(t), \widetilde{w}(t))$ evolves by Perelman's modified extended Ricci flow in $X_a \times [a,b]$, and then we are in a position to use Proposition 2.10 for the compact manifold X_a with boundary ∂X_a. So, from equation (2.19) we have on Σ_t

$$\frac{\partial}{\partial t}g_{ij}=\frac{\partial}{\partial t}\left(\left(\phi_t^*\right)^{-1}\phi_t^* g_{ij}\right)=\frac{\partial}{\partial t}\left(\left(\phi_t^*\right)^{-1}\widetilde{g}_{ij}\right)=(\phi_t^*)^{-1}\left(\frac{\partial}{\partial t}\widetilde{g}_{ij}+\left(\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^{-1}}\widetilde{g}\right)_{ij}\right)=-2(R_{ij}-\alpha_n\widehat{\nabla}_i w\widehat{\nabla}_j w)-2H\mathcal{A}_{ij},$$

which is (2.25). Likewise, from equation (2.20) one has

$$\frac{\partial}{\partial t}w=(\phi_t^*)^{-1}\left(\frac{\partial}{\partial t}\widetilde{w}+\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^{-1}}\widetilde{w}\right)=\widehat{\Delta}w+\nabla_0\nabla_0 w,$$

which is (2.26). Next, equation (2.21) implies

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{A}_{ij}&=(\phi_t^*)^{-1}\left(\frac{\partial}{\partial t}\widetilde{\mathcal{A}}_{ij}+\left(\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^{-1}}\widetilde{\mathcal{A}}\right)_{ij}\right)\\&=(\widehat{\Delta}\mathcal{A})_{ij}-\mathcal{A}^k{}_iR^l_{klj}-\mathcal{A}^k{}_jR^l_{kli}+2\mathcal{A}^{kl}R_{kilj}-2H\mathcal{A}_{ik}\mathcal{A}^k{}_j+\mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij}+\nabla_0R_{0i0j}.\end{aligned}$$

Finally, from equation (2.22) we get

$$\frac{\partial}{\partial t}H=(\phi_t^*)^{-1}\left(\frac{\partial}{\partial t}H_{\widetilde{g}}+\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^{-1}}H_{\widetilde{g}}\right)=\widehat{\Delta}H+2\mathcal{A}^{ij}R_{ij}+\mathcal{A}^{ij}\mathcal{A}_{ij}H+\nabla_0R_{00}-2\alpha_n\mathcal{A}(\widehat{\nabla}w,\widehat{\nabla}w),$$

which finishes the proof. □

Remark 2.14. We point out that equations (2.25) and (2.26) hold regardless the assumption $e_0w=0$ on Σ_0.

Remark 2.15. If M is the Euclidean space with its standard metric g_0, $g(t)=g_0$ and $w(t)=w$ is a constant, then Eqs. (2.25), (2.27) and (2.28) are the same as in Proposition 1.8.

Corollary 2.16 ([Lot12, Prop. 4])**.** *Suppose that the family $\mathscr{F}:=\{\Sigma_t\,;t\in[0,T)\}$ is a mean curvature flow in a $g(t)$-Ricci flow background. Then, the following evolution equations hold:*

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij}&=-2R_{ij}-2H\mathcal{A}_{ij},\\\frac{\partial}{\partial t}\mathcal{A}_{ij}&=(\widehat{\Delta}\mathcal{A})_{ij}-\mathcal{A}^k{}_iR^l{}_{klj}-\mathcal{A}^k{}_jR^l{}_{kli}+2\mathcal{A}^{kl}R_{kilj}-2H\mathcal{A}_{ik}\mathcal{A}^k{}_j+\mathcal{A}^{kl}\mathcal{A}_{kl}\mathcal{A}_{ij}+\nabla_0R_{0i0j}\\\frac{\partial}{\partial t}H&=\widehat{\Delta}H+2\mathcal{A}^{ij}R_{ij}+\mathcal{A}^{ij}\mathcal{A}_{ij}H+\nabla_0R_{00}.\end{aligned}$$

We can now show how the weighted extended GHY-action $I_\infty^{\alpha_n}$ change under a mean curvature flow $\{\partial M_t\}$ in an $(g(t),w(t))$-extended Ricci flow background with $e_0w=0$ on ∂M.

Theorem 2.17. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and let $I_\infty^{\alpha_n}$ be the weighted extended GHY-action on $\mathscr{P}(M)$ defined as in (2.3). Suppose that the family $\{\partial M_t\,;t\in[0,T)\}$ is a MCF in a $(g(t),w(t))$-extended Ricci flow background which satisfies $e_0w=0$ on ∂M, where e_0 is the inward unit normal vector field on ∂M. Under these* *conditions, if $u := e^{-f}$ is a solution to the conjugate heat equation*

$$\frac{\partial}{\partial t}u = -\Delta_g u + R_g u - \alpha_n |\nabla w|^2 u \tag{2.30}$$

in $M \times [0,T)$, with $e_0 u = Hu$ on ∂M, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_\infty^{\alpha_n} = 2\int_M \Big(&|\mathrm{Ric} + \mathrm{Hess}\, f - \alpha_n \mathrm{d}w \otimes \mathrm{d}w|^2 + \alpha_n \big(\Delta_g w - \langle \nabla w, \nabla f\rangle\big)^2\Big) e^{-f}\,\mathrm{dV} \\ &+ 2\int_{\partial M} \Big(\frac{\partial}{\partial t}H - 2\langle \widehat{\nabla} f, \widehat{\nabla} H\rangle + \mathcal{A}(\widehat{\nabla} f, \widehat{\nabla} f) + 2R^{0i}\widehat{\nabla}_i f - \frac{1}{2}\nabla_0 R_g - HR_{00} \\ &+ \alpha_n \mathcal{A}(\widehat{\nabla} w, \widehat{\nabla} w)\Big) e^{-f}\,\mathrm{dA}.\end{aligned}$$

In particular, if both $\big(R_{ij} + \nabla_i \nabla_j f - \alpha_n \nabla_i w \nabla_j w\big)|_{\partial M} = 0$ and $\big(R_{i0} + \nabla_i \nabla_0 f\big)|_{\partial M} = 0$ vanish on ∂M, then the boundary integrand vanishes.

Proof. The hypotheses on $\{\partial M_t\,; t \in [0,T)\}$ and on u allow us to use $\widetilde{g}(t)$, $\widetilde{f}(t)$ and $\widetilde{w}(t)$ on $M = X_a$ as in the proof of Proposition 2.13, so that $(\widetilde{g}(t), \widetilde{w}(t))$ evolves by Perelman's modified extended Ricci flow in $M \times [0,T)$. In this way, the result follows immediately from Corollary 2.11 and the fact that the identity

$$\frac{\partial}{\partial t}H_{\widetilde{g}} = \frac{\partial}{\partial t}H - \langle \widehat{\nabla} f, \widehat{\nabla} H\rangle.$$

holds on ∂M_t for all $t \in [0,T)$. □

Remark 2.18. As we pointed out in the introduction, Theorem 2.17 extends Theorem 1 in [Lot12]. Also, when M is compact without boundary, Theorem 2.17 coincides with [Lis08, Lemma 3.4].

We finish this subsection by recovering the results by Ecker [Eck07, Props. 3.2 and 3.4] and List [Lis08, Thm. 6.1].

Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in Euclidean space $\mathbb{R}^n$ with its standard metric g_0, and suppose $\{\partial\Omega_t; t \in [0,T)\}$ evolves by mean curvature flow in $(\mathbb{R}^n, g_0)$ with $\Sigma_0 = \partial\Omega$. Let $f : \Omega \to \mathbb{R}$ be a smooth function and consider the functional given by

$$\mathcal{W}(\Omega, f, \tau) = \int_\Omega \big(\tau|\nabla f|^2 + f - n\big) v\,\mathrm{dV} + 2\int_{\partial\Omega} \tau H v\,\mathrm{dA}.$$

on $\mathbb{R}^n \times C^\infty(\overline{\Omega}) \times \mathbb{R}_+$, where $v = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$, and the function $u = e^{-f}$ is the positive solution on $\bigcup_{t\in[0,T)}(\Omega_t \times \{t\}) \subset \mathbb{R}^n \times [0,T)$ of

$$\begin{cases} \frac{\partial}{\partial t}u = -\Delta u + \frac{n}{2\tau}u, \\ \frac{\mathrm{d}}{\mathrm{d}t}\tau = -1. \end{cases}$$

satisfying the boundary condition $e(\cdot,t)u = Hu$. Let $\{\phi_t\}_{t\in[0,T)}$ be the one-parameter family of diffeomorphisms generated by $\{-\nabla f(t)\}_{t\in[0,T)}$, with $\phi_0 = \mathrm{Id}$, and $\Omega_t := \phi_t(\Omega)$. By setting $\delta_{\alpha\beta}\circ\phi_t$ and $f\circ\phi_t$ we have that are defined on Ω and plugging $\frac{\partial}{\partial t}(\delta_{\alpha\beta}\circ\phi_t) = \Big((\mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t}\phi_t}\delta)_{\alpha\beta}\Big)\circ\phi_t$, $\frac{\partial}{\partial t}(f\circ\phi_t) = (-\Delta f - \frac{n}{2\tau})\circ\phi_t$ and $\xi = -1$ in the variation of $\mathcal{W}$ (see Corollary 1.21) since $\frac{\partial}{\partial t}\big((v\,\mathrm{dV})\circ\phi_t\big) = 0$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = &\int_\Omega 2\tau\Big(\Big|\nabla_\alpha\nabla_\beta f - \frac{1}{2\tau}\delta_{\alpha\beta}\Big|^2 v\Big)\circ\phi_t\,\mathrm{dV}\\ &+2\tau\int_{\partial\Omega}\Big(\big(\frac{\partial}{\partial t}H - \langle\widehat{\nabla}H,\widehat{\nabla}f\rangle + \mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f) - \frac{H}{2\tau}\big)v\Big)\circ\phi_t\,\mathrm{dA}\,.\end{aligned}$$

Since $\frac{\partial}{\partial t}(H\circ\phi_t) = \Big(\frac{\partial}{\partial t}H - \langle\widehat{\nabla}f,\widehat{\nabla}H\rangle\Big)\circ\phi_t$, we recover the following result.

Proposition 2.19 ([Eck07, Props. 3.2 and 3.4])**.**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(\Omega_t,f(t),\tau(t)) = \int_{\Omega_t}|\operatorname{Hess} f - \frac{1}{2\tau}g_0|^2 v\,\mathrm{dV} + 2\tau\int_{\partial\Omega_t}\Big(\frac{\partial}{\partial t}H - 2\langle\widehat{\nabla}H,\widehat{\nabla}f\rangle + \mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f) - \frac{H}{2\tau}\Big)v\,\mathrm{dA}\,.$$

2.3.2 Extension of Hamilton's differential Harnack expression

Here, we will see as the boundary integrand term of the time-derivative of weighted extended GHY-action provides an extension of Hamilton's differential Harnack expression for mean curvature flow in Euclidean space to the more general context of mean curvature flow in an extended Ricci flow background.

Let $\mathscr{F} := \{\Sigma_t\}$ be a family of mean curvature solitons in a $(\overline{g},\overline{w})$-extended Ricci flow background. Recall that a hypersurface $\Sigma_t\in\mathscr{F}$ is said to be a *mean curvature soliton* if

$$H(p,t) + e(p,t)\overline{f} = 0\ \forall p\in\Sigma_t.$$

Thus, the equations for the steady case

$$\overline{R}_{ij} + \overline{\nabla}_i\overline{\nabla}_j\overline{f} - \alpha_n\overline{\nabla}_i\overline{w}\overline{\nabla}_j\overline{w} = 0 \quad\text{and}\quad \overline{R}_{i0} + \overline{\nabla}_i\overline{\nabla}_0\overline{f} - \alpha_n\overline{\nabla}_i\overline{w}\overline{\nabla}_0\overline{w} = 0$$

on Σ_t become

$$\overline{R}_{ij} + \widehat{\nabla}_i\widehat{\nabla}_j\overline{f} + H_{\overline{g}}\mathcal{A}_{ij} - \alpha_n\widehat{\nabla}_i\overline{w}\widehat{\nabla}_j\overline{w} = 0, \tag{2.31}$$

and

$$\overline{R}_{i0} - \widehat{\nabla}_i H_{\overline{g}} + \mathcal{A}^k{}_i\widehat{\nabla}_k\overline{f} - \alpha_n e_0\overline{w}\widehat{\nabla}_i\overline{w} = 0. \tag{2.32}$$

Example 2.20. For instance, consider $M = \mathbb{R}^n$, $\overline{g}(t) = \delta_{\alpha\beta}$ and $\overline{w}(t) = w$ constant, and let L be a linear function on $\mathbb{R}^n$. Defining $\overline{f} = L + t|\nabla L|^2$, we have that $\overline{f}$ satisfies (2.7). Changing $\overline{f}$ to $-f$, equations (2.31) and (2.32) then become

$$\widehat{\nabla}_i\widehat{\nabla}_j f = H\mathcal{A}_{ij} \quad \text{and} \quad \widehat{\nabla}_i H + \mathcal{A}^k_i\widehat{\nabla}_k f = 0,$$

respectively, which appear in [Ham95, p. 219] as equations for a translating soliton.

Consider a bounded domain Ω with smooth boundary $\partial\Omega$ in Euclidean space $\mathbb{R}^n$, and take a solution $u = e^{-f}$ to conjugate heat equation (2.30) in $\Omega \times [0,T)$ with $e_0 u = Hu$ on $\partial\Omega$. If $\{\partial\Omega_t\,; t \in [0,T)\}$ is a mean curvature flow in a $(g(t),w(t))$-extended Ricci flow background with $g(t)$ Ricci flat and $e_0 w = 0$ on $\partial\Omega$, then the boundary integrand in Theorem 2.17 becomes

$$\mathcal{Z}(V) + \alpha_n\mathcal{A}(\widehat{\nabla}w, \widehat{\nabla}w),$$

where $V = -\widehat{\nabla}f$ and $\mathcal{Z}(V) := \frac{\partial}{\partial t}H + 2\langle V, \widehat{\nabla}H\rangle + \mathcal{A}(V,V)$ is Hamilton's differential Harnack expression for the case of mean curvature flow in Euclidean space, which vanishes in the particular case $\mathscr{F}$ is a translating solitons (cf. [Ham95, Def. 4.1 and Lem. 3.2]).

The next result suggests an extension $\mathcal{Z}^{\alpha_n}_{\overline{g},\overline{w}}$ of $\mathcal{Z}$ for the more general case of mean curvature flow in an extended Ricci flow background, whose characterization of nullity should be on the steady case. For this, we observe that, if $(\overline{g}(t), \overline{w}(t))$ is a gradient steady soliton on a smooth manifold M with potential function $\overline{f}$, and Σ is a mean curvature soliton at $t = 0$, then its ensuing mean curvature flow $\{\Sigma_t\}$ consists of mean curvature solitons, and $\{\Sigma_t\}$ differs from $\{\psi_t(\Sigma)\}$ by hypersurface diffeomorphisms. In Subsection 2.3.3, we give a more general description that includes the shrinking and expanding soliton cases.

Corollary 2.21. *Let M be an $n(\geqslant 3)$-dimensional smooth manifold, and let $(\overline{g}(t), \overline{w}(t))$ be a gradient steady soliton to the extended Ricci flow on M with potential function $\overline{f}$. Assume that $\mathscr{F} := \{\Sigma_t\,; t \in [0,T)\}$ is a mean curvature flow in a $(\overline{g},\overline{w})$-extended Ricci flow background which satisfies $H + e_0 f = 0$ and $e_0 w = 0$ on Σ_0, where e_0 is the unit normal vector field on Σ_0. Under these conditions, the differential expression*

$$\frac{\partial}{\partial t}H_{\overline{g}} - 2\langle\widehat{\nabla}_{\overline{g}}\overline{f}, \widehat{\nabla}_{\overline{g}}H\rangle_{\overline{g}} + \mathcal{A}(\widehat{\nabla}_{\overline{g}}\overline{f}, \widehat{\nabla}_{\overline{g}}\overline{f}) + 2\overline{R}^{0i}\widehat{\nabla}_i\overline{f} - \frac{1}{2}\overline{\nabla}_0 R_{\overline{g}} - H_{\overline{g}}\overline{R}_{00} + \alpha_n\mathcal{A}(\widehat{\nabla}\overline{w}, \widehat{\nabla}\overline{w})$$

vanishes on Σ_t for all $t \in [0,T)$, where $\mathcal{A}$ and and $\widehat{\nabla}_{\overline{g}}$ are as in Theorem 2.17.

Proof. If $(\overline{g}(t), \overline{w}(t))$ is a gradient steady soliton to the extended Ricci flow in $M \times [0,T)$, then the positive function $u = e^{-\overline{f}}$ on $\bigcup_{t\in[0,T)}(X_t \times \{t\}) \subset M \times [0,T)$ satisfies the conjugated heat equation (2.29) with $e_0 u = Hu$ and $e_0 w = 0$ on $\partial X_0 = \Sigma_0$, where the boundary conditions follows from the assumptions on Σ_0. To see this, first observe that $\Delta_{\overline{g}}u = (|\nabla_{\overline{g}}\overline{f}|^2_{\overline{g}} - \Delta_{\overline{g}}\overline{f})u$. Now taking traces in (2.6a) and using (2.7), we obtain

$$\frac{\partial}{\partial t}u = -u|\nabla_{\overline{g}}\overline{f}|^2_{\overline{g}} = -\Delta_{\overline{g}}u + R_{\overline{g}}u - \alpha_n|\nabla_{\overline{g}}\overline{w}|^2_{\overline{g}}u.$$

Thus, we can define $\widetilde{g}(t)$, $\widetilde{w}(t)$ and $\widetilde{f}(t)$ on X_0 as in the proof of Proposition 2.13, so that $(\widetilde{g}(t),\widetilde{w}(t))$ evolves by Perelman's modified extended Ricci flow in $X_0\times[0,T)$. Besides, again we use that $(\overline{g}(t),\overline{w}(t))$ is a gradient steady soliton and that $e_0 w = 0$ on Σ_0, to get

$$\big(\widetilde{R}_{ij} + \widetilde{\nabla}_i\widetilde{\nabla}_j\widetilde{f} - \alpha_n\widetilde{\nabla}_i\widetilde{w}\widetilde{\nabla}_j\widetilde{w}\big)|_{\Sigma_0} = 0 \quad \text{and} \quad \big(\widetilde{R}_{i0} + \widetilde{\nabla}_i\widetilde{\nabla}_0\widetilde{f}\big)|_{\Sigma_0} = 0.$$

As in the proof of Theorem 2.17, the result follows from Corollary 2.11 and the identity

$$\frac{\partial}{\partial t}H_{\widetilde{g}} = \frac{\partial}{\partial t}H_{\overline{g}} - \langle\widehat{\nabla}_{\overline{g}}\overline{f},\widehat{\nabla}_{\overline{g}}H_{\overline{g}}\rangle_{\overline{g}}.$$

This completes the proof. □

Remark 2.22. Suppose $M=\mathbb{R}^n$, $\overline{g}(t)=\delta_{\alpha\beta}$, $\overline{w}(t)=w$ constant. Let L be a linear function on $\mathbb{R}^n$ and define $\overline{f}=L+t|\nabla L|^2$. Letting $V(t)=-\widehat{\nabla}\overline{f}$, Corollary 2.21 coincides with [Ham95, Lem. 3.2].

2.3.3 Characterization of mean curvature solitons

Here, we will show how to construct a family of mean curvature solitons and establish a characterization of such a family. For it, let M be an $n(\geqslant 3)$-dimensional smooth manifold, and let $(\overline{g}(t),\overline{w}(t))$ be a gradient soliton to the extended Ricci flow on M for some initial value (g,w) and with potential function $\overline{f}=\psi_t^* f$, where $\{\psi_t\}$ is the smooth one-parameter family of diffeomorphisms of M generated by $Y_t=\nabla_g f/\sigma(t)$, with $\sigma(t)=-\kappa t$ and $\psi_{-\kappa}=\mathrm{Id}$, where $\kappa=1$ in the shrinking case (for $t\in(-\infty,0)$), $\kappa=-1$ in the expanding case (for $t\in(0,+\infty)$) and $\sigma(t)=1$ in the steady case (for $t\in\mathbb{R}$) with $\psi_0=\mathrm{Id}$ (see Proposition 2.2).

Given an $(n-1)$-dimensional smooth compact manifold Σ without boundary, let $\{x(\,\cdot\,,t)\}$ be a smooth one-parameter family of immersions of Σ into M, given by $x(\cdot,t):=\psi(\cdot,-t-2\kappa)$ and $x(\cdot,t):=\psi(\cdot,-t)$ in the steady case. Note that $x(\cdot,-\kappa)=\psi(\,\cdot\,,-\kappa)=\mathrm{Id}$ and $x(\,\cdot\,,0)=\psi(\cdot,0)=\mathrm{Id}$. Moreover, when considering $x(\cdot,t):=\psi(\cdot,-t-2\kappa)$, we are assuming $t\in(-2,0)$ in the shrinking case, $t\in(0,2)$ in the expanding case, and $t\in\mathbb{R}$ in the steady case. For each t, set $x_t=x(\,\cdot\,,t)$, Σ_t for the hypersurface $x_t(\Sigma)$ of $(M,\overline{g}(t))$, i.e., $\Sigma_t:=(x_t(\Sigma),\overline{g}(t))$, and $\mathscr{G}:=\{\Sigma_t\}$. In this setting, we prove the next two propositions. In particular, if $\mathscr{G}$ evolves by MCF in the $(\overline{g},\overline{w})$-extended Ricci flow background on M, then it is a family of mean curvature solitons. Indeed, since $\overline{g}(t)=\sigma(t)\psi_t^* g$, we have $\nabla_g f=\sigma(t)\nabla_{\overline{g}(t)}\overline{f}$, and then

$$H(p,t)=\overline{g}(t)\Big(\frac{\partial}{\partial t}x(p,t),e(p,t)\Big)=\overline{g}(t)\Big(-\frac{\nabla_g f(p)}{\sigma(t)},e(p,t)\Big)$$

$$= -\overline{g}(t)\Big(\nabla_{\overline{g}(t)}\overline{f}(p), e(p,t)\Big) = -e(p,t)\overline{f}(p),$$

it proves our claim. A sufficient condition for ensuring that $\mathscr{G}$ is a family of mean curvature solitons is that the hypersurface Σ should be f-minimal. Besides, we will see that any family $\mathscr{F}$ of mean curvature solitons is given by the family $\mathscr{G}$ up to reparametrization, as stated below.

Theorem 2.23. *If Σ is the f-minimal hypersurface of (M,g), then $\mathscr{G}$ is a family of mean curvature solitons. Moreover, any family $\mathscr{F}$ of mean curvature solitons is given by $\mathscr{G}$ up to reparametrization.*

***Proof*.** Let Σ be a hypersurface of (M,g) satisfying $H + e_o f = 0$ on Σ, where e_o is the unit normal vector field on Σ. Take $\mathscr{G} = \{\Sigma_t\}$ the smooth one-parameter family of isometric immersions of Σ into M as above, so that $e_o = \sqrt{\sigma(t)}e(\cdot,t)$, and then $\mathcal{A}_{e_o}(p) = \sqrt{\sigma(t)}\mathcal{A}_{e(p,t)}$ that implies $H(p) = \sqrt{\sigma(t)}H(p,t)$. Hence,

$$0 = H(p) + e_o f(p) = \sqrt{\sigma(t)}H(p,t) + \sqrt{\sigma(t)}e(p,t)\overline{f} = \sqrt{\sigma(t)}\big(H(p,t) + e(p,t)\overline{f}\big).$$

Thus,

$$\begin{aligned}\Big(\frac{\partial}{\partial t}x(p,t)\Big)^{\perp} &= \overline{g}(t)\Big(\frac{\partial}{\partial t}x(p,t), e(p,t)\Big)e(p,t) = \overline{g}(t)\Big(-\frac{\nabla_g f}{\sigma(t)}, e(p,t)\Big)e(p,t)\\ &= -\overline{g}(t)\Big(\nabla_{\overline{g}(t)}\overline{f}(p), e(p,t)\Big)e(p,t) = -e(p,t)(\overline{f})e(p,t) = H(p,t)e(p,t).\end{aligned}$$

Now, we affirm that if a smooth family of hypersurfaces $\Sigma_t = x_t(\Sigma)$ satisfies $\langle \frac{\partial}{\partial t}x(p,t), e(p,t)\rangle = H(p,t)$, then it can be everywhere locally reparametrized to a mean curvature flow. Indeed, if $\frac{\partial}{\partial t}x(p,t) = H(p,t)e(p,t) + X(p,t)$, where $X(p,t) \in \mathrm{d}x_t(T_p\Sigma)$ $\forall p \in \Sigma$, take $\{\varphi_t\}$ the smooth one-parameter family of diffeomorphisms of Σ generated by $Y(p,t) = -[\mathrm{d}x_t]^{-1}(X(p,t))$ and then consider the reparametrization $\widetilde{x}(p,t) = x(\varphi_t(p),t)$. By a straightforward computation $\{\widetilde{\Sigma}_t := \widetilde{x}_t(\Sigma)\}$ evolves by MCF in the $(\overline{g},\overline{w})$-extended Ricci flow background on M. Finally, by a simple analysis of this proof, we also show that any family $\mathscr{F}$ of mean curvature solitons is given by $\mathscr{G}$ up to reparametrization. □

2.3.4 Huisken-type monotonicity

It is not surprise that monotonicity formulas play an fundamental role in geometric analysis. For instance, some well-known are: Huisken's integral for the mean curvature flow (see [Hui90]), Perelman entropy formula for the Ricci flow (see [Per02]), and List entropy formula for the extended Ricci flow (cf. [Lis08]). More recently, Lott and Magni, Mantegazza and Tsatis showed monotonicity formulas for mean curvature flow when the ambient manifold moving by Ricci flow, known also as Ricci flow background (see [Lot12] and [MMT13]).

In [MMT13], Magni et al. showed some computations related to the motion by mean curvature flow of a submanifold inside an ambient Riemannian manifold evolving by Ricci or backward Ricci flow (i.e., $\frac{\partial}{\partial t}g(t) = 2\,\mathrm{Ric}_{g(t)}$). Special emphasis was given to the possible generalization of Huisken's monotonicity formula and its connection with the validity of some Li-Yau-Hamilton differential Harnack-type inequalities in a moving Riemannian manifold, as stated below.

Theorem 2.24 (Huisken's Monotonicity-type Formula [MMT13])**.** *Consider a smooth compact submanifold N^m of a Riemannian manifold $(M^n, g(t))$ evolving by the Ricci flow (or the backward Ricci flow), and $u : M^n \times [0,T) \to \mathbb{R}$ is a positive smooth solution of the backward heat equation $\frac{\partial}{\partial t}u = -\Delta_g u + Ku$, for some smooth function K on $M \times [0,T)$. If N^m moving by mean curvature in the time interval $[0,T)$, then the quantity $[4\pi(T-t)]^{\frac{n-m}{2}} \int_N u\,\mathrm{dA}$ is non increasing during the flow in $[0,T)$.*

For the proof of the previous result was essential the nonpositivity of the Li-Yau-Hamilton quantity for the Ricci flow, namely, let $(M, g(t))$ be an n-dimensional Riemannian manifold evolving by the Ricci flow $\frac{\partial}{\partial t}g = -2\,\mathrm{Ric}_g$, and let $u : M \times [0,T) \to \mathbb{R}$ be a positive solution of $\frac{\partial}{\partial t}u = -\Delta u + Ru$. Under these assumptions, considering a compact m-submanifold Σ moving by mean curvature flow, then it is true the so called Hamilton's matrix Li-Yau Harnack differential inequality

$$g^{ab}\big(\nabla_a \nabla_b f + R_{ab} - \frac{1}{2(T-t)}g_{ab}\big) \leqslant 0, \tag{2.33}$$

where $f = -\log u$, for all indices a, b associated with the coordinates which are normal to Σ.

Inspired by this work, we establish analogous results to the motion by mean curvature flow of a submanifold inside an ambient Riemannian manifold moving by extended Ricci flow.

In order to establish the main result of this section (see Theorem 2.26), we first determine how the area (i.e., the $(n-1)$-dimensional Riemannian measure) of a mean curvature flow in an extended Ricci flow background evolves.

Lemma 2.25. *Consider an $n(\geqslant 3)$-dimensional smooth manifold M and let $(\overline{g}(t), \overline{w}(t))$ be a gradient soliton to the extended Ricci flow on M with potential function $\overline{f}$. Assume that $\mathscr{F} := \{\Sigma_t\}$ is a mean curvature flow in the $(\overline{g}, \overline{w})$-extended Ricci flow background, and denote by $\mathrm{dA}_{\overline{g}}$ the $(n-1)$-dimensional Riemannian measure on Σ_t induced by $\overline{g}(t)$. Under these conditions, the following equation holds on Σ_t for all t*

$$\frac{\partial}{\partial t}(\mathrm{dA}_{\overline{g}}) = -(\overline{R}^i_{\ i} + H^2_{\overline{g}} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}})\,\mathrm{dA}_{\overline{g}}\,.$$

Proof. The lemma follows by using the well-known formula

$$\frac{\partial}{\partial t}(\mathrm{dA}_{\overline{g}}) = \frac{1}{2}\mathrm{tr}_{(\overline{g}_{ij}(t))}\Big(\frac{\partial}{\partial t}\overline{g}_{ij}\Big)\,\mathrm{dA}_{\overline{g}}$$

and equation (2.25) in Proposition 2.13 (see also Remark 2.14). $\square$

We restrict ourselves in the special case of a special solution of extended Ricci flow and hypersurfaces, more generally, see Remark 2.34.

Theorem 2.26. *Let M be an $n(\geqslant 3)$-dimensional smooth manifold. Let $(\overline{g}(t), \overline{w}(t))$ be a gradient soliton to the extended Ricci flow on M with potential function $\overline{f}$. Assume that $\mathscr{F} := \{\Sigma_t\}$ is a MCF in the $(\overline{g}, \overline{w})$-extended Ricci flow background, denote by $\mathrm{dA}_{\overline{g}}$ the $(n-1)$-dimensional Riemannian measure on Σ_t induced by $\overline{g}(t)$, and set $\mathrm{Area}_{\overline{f}}(\Sigma_t) := \int_{\Sigma_t} e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}$ Under these conditions, the function $\Phi(t)$ given by:*

(i) $\mathbb{R} \ni t \mapsto \mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the steady case;*

(ii) $(-\infty, 0) \ni t \mapsto (-t)^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the shrinking case;*

(iii) $(0, \infty) \ni t \mapsto t^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)$ *in the expanding case;*

is nonincreasing. Moreover, $\Phi(t)$ is constant if and only if $\mathscr{F}$ is a family of mean curvature solitons.

Proof. Lemma 2.25 and a straightforward computation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Area}_{\overline{f}}(\Sigma_t) = -\int_{\Sigma_t}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\overline{f} + \overline{R}^i_i + H^2_{\overline{g}} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}}\Big)e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}.$$

By Chain rule $\frac{\mathrm{d}}{\mathrm{d}t}\overline{f} = \frac{\partial}{\partial t}\overline{f}\frac{\mathrm{d}}{\mathrm{d}t}t + g(t)(\nabla_{g(t)}\overline{f}, \frac{\partial}{\partial t}x)$ that implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Area}_{\overline{f}}(\Sigma_t) = -\int_{\Sigma_t}\Big(\frac{\partial}{\partial t}\overline{f} + H_{\overline{g}}e_t\overline{f} + \overline{R}^i_i + H^2_{\overline{g}} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}}\Big)e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}.$$

First, assume $(\overline{g}(t), \overline{w}(t))$ is a gradient steady soliton. In this case, we can take traces in (2.6a) on Σ_t to get

$$0 = \overline{R}^i_i + \overline{\nabla}_i\overline{\nabla}^i\overline{f} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}} = \overline{R}^i_i + \widehat{\nabla}^i\widehat{\nabla}_i\overline{f} - H_{\overline{g}}e_t\overline{f} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}}.$$

Then, using (2.7), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Area}_{\overline{f}}(\Sigma_t) &= -\int_{\Sigma_t}\Big(|\nabla_{\overline{g}}\overline{f}|^2_{\overline{g}} - \widehat{\Delta}_{\overline{g}}\overline{f} + 2H_{\overline{g}}e_t\overline{f} + H^2_{\overline{g}}\Big)e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}\\ &= -\int_{\Sigma_t}\Big(|\widehat{\nabla}_{\overline{g}}\overline{f}|^2_{\overline{g}} + (e_t\overline{f})^2 - \widehat{\Delta}_{\overline{g}}\overline{f} + 2H_{\overline{g}}e_t\overline{f} + H^2_{\overline{g}}\Big)e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}\\ &= -\int_{\Sigma_t}\Big(H_{\overline{g}} + e_t\overline{f}\Big)^2 e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}},\end{aligned}$$

where in the second line we have used the equality

$$\widehat{\Delta}_{\overline{g}}e^{-\overline{f}} = (|\widehat{\nabla}_{\overline{g}}\overline{f}|^2_{\overline{g}} - \widehat{\Delta}_{\overline{g}}\overline{f})e^{-\overline{f}}$$

and Stokes' theorem. Since the boundary integrand in the right-hand side is nonnegative, we have immediately the result of the theorem for the steady case.

For the shrinking case, we claim that the function

$$t \in (-\infty, 0) \mapsto \tau^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)$$

is non increasing in t, where $\tau = -t$. Indeed, as above, we take traces in (2.6a) on Σ_t to obtain

$$\frac{n-1}{2\tau} = -\frac{n-1}{2t} = \overline{R}^i_{\ i} + \overline{\nabla}^i\overline{\nabla}_i\overline{f} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}} = \overline{R}^i_{\ i} + \widehat{\nabla}^i\widehat{\nabla}_i\overline{f} - H_{\overline{g}}e_t\overline{f} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}}.$$

Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\tau^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)\Big) &= -\tau^{-(n-1)/2}\int_{\Sigma_t}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\overline{f} + \overline{R}^i_{\ i} + H^2_{\overline{g}} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}}\Big)e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}\\ &\quad + \frac{(n-1)}{2}\tau^{-\frac{(n-1)}{2}-1}\int_{\Sigma_t} e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}\\ &= -\tau^{-(n-1)/2}\int_{\Sigma_t}\Big(\frac{\partial}{\partial t}\overline{f} + H_{\overline{g}}e_t\overline{f} + \overline{R}^i_{\ i} + H^2_{\overline{g}} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}} - \frac{n-1}{2\tau}\Big)e^{-\overline{f}}\mathrm{dA}_{\overline{g}}\\ &= -\tau^{-(n-1)/2}\int_{\Sigma_t}\Big(H_{\overline{g}} + e_t\overline{f}\Big)^2 e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}.\end{aligned}$$

This proves the claim, and so the theorem for the shrinking case.

Finally, in a similar way, for the expanding case, we claim that the function

$$t \in (0, +\infty) \mapsto t^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)$$

is non increasing in t. Indeed, as above, we take traces in (2.6a) on Σ_t to obtain

$$-\frac{n-1}{2t} = \overline{R}^i_{\ i} + \overline{\nabla}^i\overline{\nabla}_i\overline{f} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}} = \overline{R}^i_{\ i} + \widehat{\nabla}^i\widehat{\nabla}_i\overline{f} - H_{\overline{g}}e_t\overline{f} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}}.$$

Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(t^{-(n-1)/2}\,\mathrm{Area}_{\overline{f}}(\Sigma_t)\Big) &= -t^{-(n-1)/2}\int_{\Sigma_t}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\overline{f} + \overline{R}^i_{\ i} + H^2_{\overline{g}} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}}\Big)e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}\\ &\quad - \frac{(n-1)}{2}t^{-\frac{(n-1)}{2}-1}\int_{\Sigma_t} e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}\\ &= -t^{-(n-1)/2}\int_{\Sigma_t}\Big(\frac{\partial}{\partial t}\overline{f} + H_{\overline{g}}e_t\overline{f} + \overline{R}^i_{\ i} + H^2_{\overline{g}} - \alpha_n|\widehat{\nabla}_{\overline{g}}\overline{w}|^2_{\overline{g}}\Big)e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}\\ &= -t^{-(n-1)/2}\int_{\Sigma_t}\Big(H_{\overline{g}} + e_t\overline{f}\Big)^2 e^{-\overline{f}}\,\mathrm{dA}_{\overline{g}}.\end{aligned}$$

This completes the proof of the theorem. $\square$

Remark 2.27. For the shrinking case in Theorem 2.26, we can recover Huisken's monotonicity formula [Hui90, Thm. 3.1], by taking $M=\mathbb{R}^n$, $g_{\alpha\beta}(\tau)=\delta_{\alpha\beta}$, $\overline{f}(x,\tau)=\frac{|x|^2}{4\tau}$ and $\overline{w}(\tau)=w$ constant.

To obtain a generalization of Theorem 2.26 it is enough to prove an analogue of Li-Yau-Hamilton Harnack differential inequality (2.33) in a moving ambient space. More precisely, we need of an extension of Li-Yau-Hamilton Harnack differential inequality for the extended Ricci flow case, namely, let $(M,g(t),w(t))$ be an $n(\geqslant 3)$-dimensional Riemannian manifold evolving by the extended Ricci flow $\frac{\partial}{\partial t}g=-2\,\mathrm{Ric}_g+2\alpha_n\,\mathrm{d}w\otimes\mathrm{d}w$ with $\frac{\partial}{\partial t}w=\Delta_g w$, and let $u:M\times[0,T)\to\mathbb{R}$ be a positive solution of $\frac{\partial}{\partial t}u=-\Delta u+Ru-\alpha_n|\nabla w|^2u$. Under these assumptions, considering a compact m-submanifold Σ moving by mean curvature flow, we are assuming that

$$g^{ab}\big(R_{ab}+\nabla_a\nabla_b f-\alpha_n\nabla_a w\nabla_b w-\frac{1}{2(T-t)}g_{ab}\big)\leqslant 0 \tag{2.34}$$

where $f=-\log u$, for all indices a,b associated with the coordinates which are normal to Σ, so that we can obtain the following generalization of Theorem 2.26.

Theorem 2.28. *Consider an $n(\geqslant 3)$-dimensional smooth manifold M and let $(g(t),w(t))$ be an extended Ricci flow on $M\times[0,T)$, and $u:M\times[0,T)\to\mathbb{R}$ is a positive smooth solution of the backward heat equation $\frac{\partial}{\partial t}u=-\Delta_g u+Ku$, where $\mathrm{K}=R_g-\alpha_n|\nabla w|^2$. Assume that $\mathscr{F}:=\{\Sigma_t\,;t\in[0,T)\}$ is a mean curvature flow in the $(g(t),w(t))$-extended Ricci flow background and holds an extension of Li-Yau-Hamilton Harnack differential inequality (2.34). Under these conditions, the function*

$$t\in(0,T]\mapsto[4\pi(T-t)]^{\frac{n-m}{2}}\int_{\Sigma_t}u\,\mathrm{dA}_{g(t)},$$

is non increasing.

Proof. Indeed, consider $\mathrm{Q}=\mathrm{Ric}_g-\alpha_n\,\mathrm{d}w\otimes\mathrm{d}w$ and $\tau=T-t$ in [MMT13, Sect. 2] so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\tau^{\frac{n-m}{2}}\int_\Sigma u\,\mathrm{dA}\Big\}=&-\tau^{\frac{n-m}{2}}\int_\Sigma|H+\nabla^\perp f|^2e^{-f}\,\mathrm{dA}\\&+\tau^{\frac{n-m}{2}}\int_\Sigma g^{ab}(\nabla_a\nabla_b f+\mathrm{Q}_{ab}-\frac{1}{2\tau}g_{ab})e^{-f}\,\mathrm{dA}\\&+\tau^{\frac{n-m}{2}}\int_\Sigma(\mathrm{K}-\mathrm{tr}\,\mathrm{Q})e^{-f}\,\mathrm{dA}\,.\end{aligned}$$

Consequently, the required generalization follows from (2.34) and $\mathrm{K}=\mathrm{tr}\,\mathrm{Q}$. So, inequality (2.34) is clearly a stronger property, and then, worthy of a future work. □

2.4 Examples of solitons solution to the extended Ricci flow

In this section, we show how to obtain a gradient soliton solution to the extended Ricci flow, and then we are obtaining its corresponding extended Ricci flow and explicit examples of mean curvature flow in an extended Ricci flow background (see Propositions 2.1 and 2.2 and Theorem 2.23).

Let $g = \frac{1}{F^2} g_0$ be a Riemannian metric on $\mathbb{R}^n$, where g_0 stands for the Euclidean metric and F is a nonzero smooth function on $\mathbb{R}^n$, and consider

$$\begin{cases} \mathrm{Ric}_g + \mathrm{Hess}_g f - \alpha_n \, dw \otimes dw = \lambda g, & (2.35a) \\ \Delta_g w = \langle \nabla_g f, \nabla_g w \rangle_g. & (2.35b) \end{cases}$$

Since the metric g is conformal to g_0, we have

$$\mathrm{Ric}_g = \frac{1}{F^2}\Big((n-2)F \,\mathrm{Hess}\, F + (F\Delta F - (n-1)|\nabla F|^2) g_0\Big)$$

and the following equations are valid

$$\begin{aligned} (\mathrm{Hess}_g h)_{ij} &= h_{x_i x_j} + \frac{F_{x_j}}{F} h_{x_i} + \frac{F_{x_i}}{F} h_{x_j} \quad \text{for} \quad \mathrm{i} \neq \mathrm{j}, \\ (\mathrm{Hess}_g h)_{ii} &= h_{x_i x_i} + 2\frac{F_{x_i}}{F} h_{x_i} - \sum_k \frac{F x_k}{F} h_{x_k} \quad \text{for} \quad \mathrm{i} = \mathrm{j}, \end{aligned}$$

for any smooth function h on $\mathbb{R}^n$, see e.g. Besse [Bes07]. Hence,

$$\Delta_g h = F^2\big(\sum_k h_{x_k x_k} + (2-n)\frac{1}{F}\sum_k F_{x_k} h_{x_k}\big).$$

We find solutions of Eq. (2.35a) (as well (2.35b)) of the form $f(\xi)$ and $w(\xi)$, that is, they only depend on $\xi = \sum_{i=1}^n \alpha_i x_i$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^n \alpha_i^2 = 1$. The following proposition provides the system of ordinary differential equations that must be satisfied by such solutions, and then we can obtain all parameters necessary to construct gradient soliton solutions to extended Ricci flow on $\mathbb{R}^n$.

Proposition 2.29. *Let $\mathbb{R}^n$, with $n \geqslant 3$, be an Euclidean space with coordinates $x = (x_1, \ldots, x_n)$ and metric $g = \frac{1}{F^2(\xi)} g_0$, where $F(\xi) \in C^\infty(\mathbb{R}^n)$, $\xi = \sum_{i=1}^n \alpha_i x_i$ with $\alpha_i \in \mathbb{R}$. We can obtain smooth functions $f(\xi)$ and $w(\xi)$ satisfying (2.35a) (as well (2.35b)) by means of the equation*

$$\frac{F''}{F} - (n-1)\Big(\frac{F'}{F}\Big)^2 + \frac{1}{n-2} f'^2 - \frac{w'' f'}{(n-2)w'} = \frac{\lambda}{F^2}. \qquad (2.36)$$

Proof. We need to analyze Eq. (2.35a) in two cases. For $i \neq j$, it rewrites as

$$(n-2)\frac{F_{x_ix_j}}{F} + f_{x_ix_j} + \frac{F_{x_j}}{F} f_{x_i} + \frac{F_{x_i}}{F} f_{x_j} - \alpha_n w_{x_i} w_{x_j} = 0, \tag{2.37}$$

and for $i = j$,

$$(n-2)\frac{F_{x_ix_i}}{F} + \sum_k \frac{F_{x_kx_k}}{F} - \frac{n-1}{F^2}\sum_k F_{x_k}^2 + f_{x_ix_i} + 2\frac{F_{x_i}}{F} f_{x_i} - \sum_k \frac{F_{x_k}}{F} f_{x_k} - \alpha_n w_{x_i}^2 = \frac{\lambda}{F^2}. \tag{2.38}$$

On the other hand, equation (2.35b) becomes

$$F^2\big(\sum_k w_{x_kx_k} + (2-n)\frac{1}{F}\sum_k F_{x_k} w_{x_k}\big) = F^2 \sum_k f_{x_k} w_{x_k}. \tag{2.39}$$

We now assume that the argument ξ of the functions $F(\xi)$, $f(\xi)$ and $w(\xi)$ is of the form $\xi = \sum_{i=1}^n \alpha_i x_i$. Hence, we have $F_{x_i} = F'\alpha_i$ and $F_{x_ix_j} = F''\alpha_i\alpha_j$ where the superscript $'$ denotes the derivative with respect to ξ. Using the same reasoning for f and w, equations (2.37) and (2.38) become

$$(n-2)\frac{F''}{F} + f'' + 2\frac{F'}{F} f' - \alpha_n {w'}^2 = 0 \tag{2.40}$$

and

$$\begin{aligned}(n-2)\frac{F''}{F}\alpha_i^2 + \sum_k \frac{F''}{F}\alpha_k^2 - \frac{n-1}{F^2}\sum_k {F'}^2\alpha_k + f''\alpha_i^2 + 2\frac{F'}{F} f'\alpha_i^2 - \sum_k \frac{F'}{F} f'\alpha_k^2 - \alpha_n {w'}^2\alpha_i^2 \\ = \frac{\lambda}{F^2}.\end{aligned} \tag{2.41}$$

Since $n \geqslant 3$, we can choose this invariance so that at least two indices i, j are such that $\alpha_i\alpha_j \neq 0$ and $\sum_{i=1}^n \alpha_i^2 = 1$, and then equations (2.40) and (2.41) become

$$(n-2)\frac{F''}{F} + f'' + 2\frac{F'}{F} f' - \alpha_n {w'}^2 = 0 \tag{2.42}$$

and

$$(n-2)\frac{F''}{F}\alpha_i^2 + \frac{F''}{F} - (n-1)\Big(\frac{F'}{F}\Big)^2 + f''\alpha_i^2 + 2\frac{F'}{F} f'\alpha_i^2 - \frac{F'}{F} f' - \alpha_n {w'}^2\alpha_i^2 = \frac{\lambda}{F^2}. \tag{2.43}$$

Plugging (2.42) into (2.43), one has

$$\frac{F''}{F} - (n-1)\Big(\frac{F'}{F}\Big)^2 - \frac{F'}{F} f' = \frac{\lambda}{F^2}. \tag{2.44}$$

Eq. (2.39) provides $w'' - (n-2)\frac{F'}{F}w' = f'w'$. Assuming $w' \neq 0$ and using (2.44), we obtain

$$\frac{F''}{F} - (n-1)\left(\frac{F'}{F}\right)^2 + \frac{1}{n-2}f'^2 - \frac{w''f'}{(n-2)w'} = \frac{\lambda}{F^2}.$$

This finishes the proof of the proposition. □

We also find radial solutions of Eq. (2.35a) (as well (2.35b)) of the form $f(r)$ and $w(r)$, that is, they only depend on $r = \|x\|^2$ with $x \in \mathbb{R}^n$.

Proposition 2.30. *Let $\mathbb{R}^n$, with $n \geqslant 3$, be an Euclidean space with coordinates $x = (x_1, \dots, x_n)$ and metric $g = \frac{1}{F^2(r)}g_0$, where $F(r) \in C^\infty(\mathbb{R}^n)$, $r = \|x\|^2$. We can obtain smooth functions $f(r)$ and $w(r)$ satisfying (2.35a) (as well (2.35b)) by means of the equation*

$$2f' + 4\left[(n-1)\frac{F'}{F} + \frac{F''}{F}r - (n-1)\left(\frac{F'}{F}\right)^2 + \frac{1}{(n-2)}\left(f'r - 4\frac{w''}{w'} - 2n\right)f'\right] = \frac{\lambda}{F^2}. \tag{2.45}$$

Proof. Since $r = \|x\|^2$, we have $F_{x_i} = 2F'x_i$ and $F_{x_ix_j} = 4F''x_ix_j$ where the superscript $'$ denotes the derivative with respect to r, for all $i \neq j$. Besides, $F_{x_i} = 2F'x_i$ and $F_{x_ix_i} = 4F''x_i^2 + 2F'$, for all $i = j$. Using the same reasoning for f and w, equations (2.37) and (2.38) become

$$4x_ix_j\left[(n-2)\frac{F''}{F} + f'' + 2\frac{F'}{F}f' - \alpha_n w'^2\right] = 0 \tag{2.46}$$

and

$$\begin{aligned}4x_i^2&\left[(n-2)\frac{F''}{F} + f'' + 2\frac{F'}{F}f' - \alpha_n w'^2\right] + 2f' + 4\left[(n-1)\frac{F'}{F} + \frac{F''}{F}r - (n-1)\left(\frac{F'}{F}\right)^2 - \frac{F'}{F}f'r\right]\\ &= \frac{\lambda}{F^2}.\end{aligned} \tag{2.47}$$

Plugging (2.46) into (2.47), one has

$$2f' + 4\left[(n-1)\frac{F'}{F} + \frac{F''}{F}r - (n-1)\left(\frac{F'}{F}\right)^2 - \frac{F'}{F}f'r\right] = \frac{\lambda}{F^2}. \tag{2.48}$$

Eq. (2.39) provides $4w'' + 2nw' - (n-2)\frac{1}{F}F'w'r = f'w'r$. Assuming $w' \neq 0$ and using (2.44), we obtain

$$2f' + 4\left[(n-1)\frac{F'}{F} + \frac{F''}{F}r - (n-1)\left(\frac{F'}{F}\right)^2 + \frac{1}{(n-2)}\left(f'r - 4\frac{w''}{w'} - 2n\right)f'\right] = \frac{\lambda}{F^2}.$$

□

Remark 2.31. For constructing a family of mean curvature solitons $\mathscr{G}$ in the $(\overline{g}, \overline{w})$-extended Ricci flow background on $(\mathbb{R}^n, \frac{1}{F^2}g_0)$, it is enough to consider a f-minimal hypersurface Σ in this geometric ambient space. Indeed, it follows immediately from Propositions 2.1 and 2.2, Theorem 2.23 and Propositions 2.29 and 2.30.

In what follows, we are using Propositions 2.29 and 2.30 to show how to obtain explicit parameter functions for constructing gradient soliton solutions to the extended Ricci flow.

Example 2.32. We consider the conformal factor $F(r)$, where $r = \|x\|^2$, given by $F(r) = e^{-\frac{r^2}{2}}$ and the potential of the gaussian soliton, $f(r) = cr$, $c \neq 0$. From equation (2.45) we obtain

$$2c+4\Big[-(n-1)r+r^2-1-(n-1)r^2+\frac{1}{(n-2)}\Big(cr-4\frac{w''}{w'}-2n\Big)c\Big] = \lambda e^{r^2}.$$

Therefore

$$\frac{w''}{w'} = \frac{n-2}{4c}\Big\{2c+4\Big[-(n-1)r+r^2-1-(n-1)r^2+\frac{1}{(n-2)}\Big(cr-2n\Big)c\Big]-\lambda e^{r^2}\Big\},$$

whence

$$\ln w' = \frac{n-2}{4c}\Big\{2cr+4\Big[-(n-1)\frac{r^2}{2}+\frac{r^3}{3}-1-(n-1)\frac{r^3}{3}+\frac{1}{(n-2)}\Big(c\frac{r^2}{2}-2nr\Big)c\Big]-\lambda\int e^{r^2}\,\mathrm{d}r\Big\}+c_1,$$

for some constant c_1. Hence

$$w' = e^{\Big\{\frac{n-2}{4c}\Big[2cr+4\Big(-(n-1)\frac{r^2}{2}+\frac{r^3}{3}-1-(n-1)\frac{r^3}{3}+\frac{1}{(n-2)}\Big(c\frac{r^2}{2}-2nr\Big)c\Big)-\lambda\int e^{r^2}\,\mathrm{d}r\Big]+c_1\Big\}}$$

and then

$$w = \int e^{\Big\{\frac{n-2}{4c}\Big[2cr+4\Big(-(n-1)\frac{r^2}{2}+\frac{r^3}{3}-1-(n-1)\frac{r^3}{3}+\frac{1}{(n-2)}\Big(c\frac{r^2}{2}-2nr\Big)c\Big)-\lambda\int e^{r^2}\,\mathrm{d}r\Big]+c_1\Big\}}\,\mathrm{d}r.$$

Example 2.33. For $f(\xi) = e^{\xi}$ and $F(\xi) = e^{-\xi}$, from equation (2.36) we have

$$1-(n-1)+\frac{e^{2\xi}}{n-2}-\frac{w''e^{\xi}}{(n-2)w'} = \lambda e^{2\xi}.$$

Therefore

$$\frac{w''}{w'} = -(n-2)^2e^{-\xi}-\lambda(n-2)e^{\xi}+e^{\xi}, \quad \text{and then} \quad \mathrm{\ln w' = (n-2)^2e^{-\xi}-\lambda(n-2)e^{\xi}+e^{\xi}+c},$$

for some constant c. Hence

$$w' = e^{[(n-2)^2e^{-\xi}-\lambda(n-2)e^{\xi}+e^{\xi}+c]} \quad \text{and then} \quad \mathrm{w = \int e^{[(n-2)^2e^{-\xi}-\lambda(n-2)e^{-\xi}+e^{\xi}+c]}\,d\xi}.$$

Example 2.34. For $f(\xi) = \mathrm{tg}\xi$ and $F(\xi) = \mathrm{cotg}\xi$, with $0 < \xi < \frac{\pi}{2}$, from equation (2.36) we have

$$2\mathrm{cossec}\xi-\frac{4(n-1)}{\sin^2 2\xi}+\frac{\sec^4\xi}{n-2}-\frac{w''\sec^2\xi}{(n-2)w'} = \lambda\mathrm{tg}^2\xi.$$

Therefore,

$$\frac{w'' \sec^2 \xi}{(n-2)w'} = 2\mathrm{cossec}\xi - \frac{4(n-1)}{\sin^2 2\xi} + \frac{\sec^4 \xi}{n-2} - \lambda \mathrm{tg}^2 \xi.$$

Hence,

$$\begin{aligned}\frac{w''}{w'} &= \frac{n-2}{\sec^2 \xi}\Big(2\mathrm{cossec}\xi - \frac{4(n-1)}{\sin^2 2\xi} + \frac{\sec^4 \xi}{n-2} - \lambda \mathrm{tg}^2 \xi\Big)\\ &= 2(n-2)\mathrm{cotg}\xi - (n-1)(n-2)\mathrm{cossec}\xi + \sec^2 \xi - \lambda(n-2)\sin^2 \xi.\end{aligned}$$

Whence,

$$\ln w' = \int \Big(2(n-2)\mathrm{cotg}\xi - (n-1)(n-2)\mathrm{cossec}\xi + \sec^2 \xi - \lambda(n-2)\sin^2 \xi\Big)\,\mathrm{d}\xi + c,$$

for some constant c. Thus,

$$\begin{aligned}w' &= \exp\Big\{\int \Big(2(n-2)\mathrm{cotg}\xi - (n-1)(n-2)\mathrm{cossec}\xi + \sec^2 \xi - \lambda(n-2)\sin^2 \xi\Big)\,\mathrm{d}\xi + c\Big\}\\ &= \exp\Big\{2(n-2)\ln\sin\xi - (n-1)(n-2)\big(\ln\sin\frac{\xi}{2} - \ln\cos\frac{\xi}{2}\big) + \mathrm{tg}\xi - \lambda(n-2)\big(\frac{1}{2}\xi\\ &\quad - \frac{1}{2}\sin\xi\cos\xi\big) + c\Big\}.\end{aligned}$$

So,

$$\begin{aligned}w = \int \exp\Big\{&2(n-2)\ln\sin\xi - (n-1)(n-2)\big(\ln\sin\frac{\xi}{2} - \ln\cos\frac{\xi}{2}\big) + \mathrm{tg}\xi - \lambda(n-2)\big(\frac{1}{2}\xi\\ &- \frac{1}{2}\sin\xi\cos\xi\big) + c\Big\}\,\mathrm{d}\xi.\end{aligned}$$

Example 2.35. Let $\mathbb{B}^n_+ \subset \mathbb{R}^n$, $n \geqslant 3$, be a unitary upper half ball with metric $g = \frac{1}{(1+x_n)^2} g_0$. Note that its boundary is the standard unitary sphere $(\mathbb{S}^{n-1}, g_0)$, $\xi = x_n$ and $F(x_n) = 1 + x_n$. Moreover, the mean curvature of $(\mathbb{S}^{n-1}, g_0)$ with respect to $e_0 = -e_n$ is $H_{g_0} = n-1$, so that we can take $f(x) = (n-1)\langle x, e_n\rangle = (n-1)x_n$, since $H_{g_0} + e_0 f = 0$. By Proposition 2.29,

$$\frac{F''}{F} - (n-1)\Big(\frac{F'}{F}\Big)^2 + \frac{1}{n-2} f'^2 - \frac{w'' f'}{(n-2)w'} = \frac{\lambda}{F^2}.$$

Since $F' = 1$ and $F'' = 0$, we get

$$\frac{(n-1)w''}{(n-2)w'} = -\frac{n-1}{(1+x_n)^2} + \frac{(n-1)^2}{n-2} - \frac{\lambda}{(1+x_n)^2}.$$

So,

$$\frac{w''}{w'} = -\frac{n-2}{(1+x_n)^2} + n - 1 - \frac{\lambda(n-2)}{(n-1)(1+x_n)^2}.$$

Whence,

$$\ln w' = \frac{n-2}{1+x_n} + (n-1)x_n + \frac{\lambda(n-2)}{(n-1)(1+x_n)} + c,$$

for some constant c, that implies

$$w' = e^{\left\{\frac{n-2}{1+x_n} + (n-1)x_n + \frac{\lambda(n-2)}{(n-1)(1+x_n)} + c\right\}},$$

and then

$$w = \int e^{\left\{\frac{n-2}{1+x_n} + (n-1)x_n + \frac{\lambda(n-2)}{(n-1)(1+x_n)} + c\right\}} \mathrm{d}x_n.$$

Chapter 3

Perelman's Entropy-type

Let M an $n(\geqslant 3)$-dimensional compact smooth manifold without boundary. In [Lis08], List defined the $\mathcal{W}$*-type* functional on $\mathscr{P}(M)\times\mathbb{R}_+$ by

$$\mathcal{W}^{\alpha_n}(g,f,w,\tau)=\int_M[\tau(R_\infty-\alpha_n|\nabla w|^2)+f-n]u\,\mathrm{dV},$$

where $u:=(4\pi\tau)^{-\frac{n}{2}}e^{-f}$.

Suppose now M is an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and consider a *Perelman's Entropy-type* on $\mathscr{P}(M)\times\mathbb{R}_+$ given by

$$\mathcal{W}^{\alpha_n}_\infty(g,f,w,\tau)=\int_\Omega[\tau(R_\infty-\alpha_n|\nabla w|^2)+f-n]u\,\mathrm{dV}+2\int_{\partial\Omega}\tau H_\infty u\,\mathrm{dA}\,. \tag{3.1}$$

In this chapter, we deal specifically with this functional.

Theorem 3.1. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and let $\mathcal{W}^{\alpha_n}_\infty$ be the Perelman's Entropy-type on $\mathscr{P}(M)\times\mathbb{R}_+$ defined in (3.1). Its evolution is given by*

$$\begin{aligned}
&\delta\mathcal{W}^{\alpha_n}_\infty(v,h,\vartheta,\xi)\\
&=\int_M\Big[\big(\xi g^{\alpha\beta}-\tau v^{\alpha\beta}\big)\Big(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f-\alpha_n\nabla_\alpha w\nabla_\beta w-\frac{1}{2\tau}g_{\alpha\beta}\Big)+2\tau\alpha_n\vartheta(\Delta_g w-\langle\nabla f,\nabla w\rangle)\\
&\quad+\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}\Big)\Big(R_g-|\nabla f|^2+2\Delta_g f+\frac{f-n-1}{\tau}-\alpha_n|\nabla w|^2\Big)\Big]u\,\mathrm{dV}\\
&\quad+\int_{\partial M}\Big[\big(\xi(2H+e_0f)+2\alpha_n\tau\vartheta e_0w\big)-\tau\big(\mathcal{A}^{ij}v_{ij}+v^{00}e_0f\big)+2\tau\Big(\frac{v^\alpha{}_\alpha}{2}-h-\frac{n\xi}{2\tau}\Big)e_0f\\
&\quad+2\tau H\Big(\frac{g^{ij}v_{ij}}{2}-h-\frac{n\xi}{2\tau}\Big)\Big]u\,\mathrm{dA},
\end{aligned}$$

where $u:=(4\pi\tau)^{-\frac{n}{2}}e^{-f}$.

Proof. The proof is very similar to the proof of the corresponding proposition of Chapter 1, see

Proposition 1.20. Observe that the functional in (3.1) can be decomposed as

$$\mathcal{W}_\infty^{\alpha_n}(g,f,w,\tau) = \mathcal{W}_\infty(g,f,\tau) - \frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\int_M \alpha_n |\nabla w|^2 e^{-f}\,\mathrm{dV}\,.$$

Moreover, we can calculate the variation $\delta\mathcal{W}_\infty^{\alpha_n}$ at (g,f,w,τ) in the direction of (v,h,ϑ,ξ) as follows

$$\delta\mathcal{W}_\infty^{\alpha_n}(v,h,\vartheta,\xi) = \delta\mathcal{W}_\infty^{\alpha_n}(v,h,0,\xi) + \delta\mathcal{W}_\infty^{\alpha_n}(0,0,\vartheta,0).$$

So,

$$\begin{aligned}\delta\mathcal{W}_\infty^{\alpha_n}(v,h,\vartheta,\xi) = \delta\mathcal{W}_\infty(v,h,\xi) - \delta\Big(\frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\int_M \alpha_n|\nabla w|^2 e^{-f}\,\mathrm{dV}\Big)(v,h,0,\xi)\\ - \frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\delta\Big(\int_M \alpha_n|\nabla w|^2 e^{-f}\,\mathrm{dV}\Big)(0,0,\vartheta).\end{aligned}$$

We will now compute each of the terms in the previous equality. The first one of them is (see Proposition 1.20)

$$\begin{aligned}\delta\mathcal{W}_\infty(v,h,\xi) = \int_M (\xi g^{\alpha\beta} - \tau v^{\alpha\beta})(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f - \frac{1}{2\tau}g_{\alpha\beta}) + \tau\Big(\frac{v^\alpha{}_\alpha}{2} - h - \frac{n\xi}{2\tau}\Big)\Big(R_g - |\nabla f|^2\\ + 2\Delta_g f + \frac{f-n-1}{\tau}\Big)u\,\mathrm{dV} + \int_{\partial M}\Big[\xi(2H + e_0 f) - \tau\big(\mathcal{A}^{ij}v_{ij} + v^{00}e_0 f\big)\\ + 2\tau\Big(\frac{v^\alpha{}_\alpha}{2} - h - \frac{n\xi}{2\tau}\Big)e_0 f + 2\tau H\Big(\frac{g^{ij}v_{ij}}{2} - h - \frac{n\xi}{2\tau}\Big)\Big]u\,\mathrm{dA}\,.\end{aligned}$$

The second one is

$$\begin{aligned}&\delta\Big(\frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\int_M \alpha_n|\nabla w|^2 e^{-f}\,\mathrm{dV}\Big)(v,h,0,\xi)\\ &= \Big(1 - \frac{n}{2}\Big)\frac{\xi}{(4\pi\tau)^{\frac{n}{2}}}\int_M \alpha_n|\nabla w|^2 e^{-f}\,\mathrm{dV} + \frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\int_M \alpha_n\Big(-v^{\alpha\beta}\nabla_\alpha w\nabla_\beta w + |\nabla w|^2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)\Big)e^{-f}\,\mathrm{dV}\\ &= \int_M \big(1 - \frac{n}{2}\big)\xi\,\alpha_n|\nabla w|^2 u\,\mathrm{dV} + \int_M \tau\alpha_n\Big(-v^{\alpha\beta}\nabla_\alpha w\nabla_\beta w + |\nabla w|^2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)\Big)u\,\mathrm{dV}\,.\end{aligned}$$

The third one is

$$\begin{aligned}\frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\delta\Big(\int_M \alpha_n|\nabla w|^2 e^{-f}\,\mathrm{dV}\Big)(0,0,\vartheta) = \int_M \tau\alpha_n\Big(-2\vartheta\Delta_g w + 2\vartheta\langle\nabla f,\nabla w\rangle\Big)u\,\mathrm{dV}\\ - 2\alpha_n\tau\int_{\partial M}\vartheta e_0 w u\,\mathrm{dA}\,.\end{aligned}$$

Putting this together gives the variation

$$\delta\mathcal{W}_\infty^{\alpha_n}(v,h,\vartheta,\xi)$$

$$
\begin{aligned}
&= \int_M \Big[(\xi g^{\alpha\beta} - \tau v^{\alpha\beta})(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f - \frac{1}{2\tau} g_{\alpha\beta}) + 2\tau\alpha_n\vartheta(\Delta_g w - \langle\nabla f, \nabla w\rangle) + \tau\Big(\frac{v^\alpha{}_\alpha}{2} - h \\
&\quad - \frac{n\xi}{2\tau}\Big)\Big(R_g - |\nabla f|^2 + 2\Delta_g f + \frac{f-n-1}{\tau}\Big) - (1 - \frac{n}{2})\xi\alpha_n|\nabla w|^2 + \tau\alpha_n v^{\alpha\beta}\nabla_\alpha w\nabla_\beta w \\
&\quad - \alpha_n\tau|\nabla w|^2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)\Big] u\, \mathrm{dV} + \int_{\partial M}\Big[\xi(2H + e_0 f) + 2\alpha_n\tau\vartheta e_0 w - \tau\Big(v^{ij}\mathcal{A}_{ij} + v^{00}e_0 f\Big) \\
&\quad + 2\tau\Big(\frac{v^\alpha{}_\alpha}{2} - h - \frac{n\xi}{2\tau}\Big)e_0 f + 2\tau H\Big(\frac{g^{ij}v_{ij}}{2} - h - \frac{n\xi}{2\tau}\Big)\Big] u\, \mathrm{dA}\,.
\end{aligned}
$$

Absorb $\tau\alpha_n v^{\alpha\beta}\nabla_\alpha w\nabla_\beta w$ into the second bracket of the terms on the first line to get

$$
\begin{aligned}
&\delta\mathcal{W}_\infty^{\alpha_n}(v, h, \vartheta, \xi) \\
&= \int_M \Big[(\xi g^{\alpha\beta} - \tau v^{\alpha\beta})(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f - \alpha_n\nabla_\alpha w\nabla_\beta w - \frac{1}{2\tau} g_{\alpha\beta}) + 2\tau\alpha_n\vartheta(\Delta_g w - \langle\nabla f, \nabla w\rangle) \\
&\quad + \tau\Big(\frac{v^\alpha{}_\alpha}{2} - h - \frac{n\xi}{2\tau}\Big)\Big(R_g - |\nabla f|^2 + 2\Delta_g f + \frac{f-n-1}{\tau}\Big) + \frac{n}{2}\xi\alpha_n|\nabla w|^2 \\
&\quad - \alpha_n\tau|\nabla w|^2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)\Big] u\, \mathrm{dV} + \int_{\partial M}\Big[\xi(2H + e_0 f) + 2\alpha_n\tau\vartheta e_0 w - \tau\Big(v^{ij}\mathcal{A}_{ij} + v^{00}e_0 f\Big) \\
&\quad + 2\tau\Big(\frac{v^\alpha{}_\alpha}{2} - h - \frac{n\xi}{2\tau}\Big)e_0 f + 2\tau H\Big(\frac{g^{ij}v_{ij}}{2} - h - \frac{n\xi}{2\tau}\Big)\Big] u\, \mathrm{dA}\,.
\end{aligned}
$$

The required integral formula follows from absorbing $-\alpha_n\tau|\nabla w|^2\Big(\frac{v^\alpha{}_\alpha}{2} - h\Big)$ into the second bracket of the terms on the second line. $\square$

Corollary 3.2. *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold with boundary ∂M, and let $\mathcal{W}_\infty^{\alpha_n}$ be Perelman's Entropy-type on $\mathscr{P}(M) \times \mathbb{R}_+$ defined in (3.1). If $\frac{v^\alpha{}_\alpha}{2} - h - \frac{n\xi}{2\tau} = 0$ on M, then its evolution is given by*

$$
\begin{aligned}
&\delta\mathcal{W}_\infty^{\alpha_n}(v, h, \vartheta, \xi) \\
&= \int_M \Big[(\xi g^{\alpha\beta} - \tau v^{\alpha\beta})\big(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f - \alpha_n\nabla_\alpha w\nabla_\beta w - \frac{1}{2\tau} g_{\alpha\beta}\big) + 2\tau\alpha_n\vartheta(\Delta_g w - \langle\nabla f, \nabla w\rangle)\Big] u\, \mathrm{dV} \\
&\quad + \int_{\partial M}\Big[\xi(2H + e_0 f) + 2\alpha_n\tau\vartheta e_0 w\Big] u\, \mathrm{dA} - \int_{\partial M}\tau\Big(\mathcal{A}^{ij}v_{ij} + v^{00}(H + e_0 f)\Big) u\, \mathrm{dA}\,.
\end{aligned}
$$

Proof. Follows from Theorem 3.1 and equation (1.34). $\square$

Corollary 3.3 ([Lis08, Sect. 6])**.** *Let M be an $n(\geqslant 3)$-dimensional compact smooth manifold without boundary, and let $\mathcal{W}_\infty^{\alpha_n}$ be Perelman's Entropy-type on $\mathscr{P}(M) \times \mathbb{R}_+$ defined as in (3.1). If $\frac{v^\alpha{}_\alpha}{2} - h - \frac{n\xi}{2\tau} = 0$ on M, then its evolution is given by*

$$
\begin{aligned}
\delta\mathcal{W}_\infty^{\alpha_n}(v, h, \vartheta, \xi) = \int_M \Big[&(\xi g^{\alpha\beta} - \tau v^{\alpha\beta})\big(R_{\alpha\beta} + \nabla_\alpha\nabla_\beta f - \alpha_n\nabla_\alpha w\nabla_\beta w - \frac{1}{2\tau} g_{\alpha\beta}\big) \\
&+ 2\tau\alpha_n\vartheta(\Delta_g w - \langle\nabla f, \nabla w\rangle)\Big] u\, \mathrm{dV}\,.
\end{aligned}
$$

In the line of Corollary 3.3, List [Lis08, Thm. 6.1] showed that the system

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2(\mathrm{Ric}_{g(t)} + \mathrm{Hess}_{g(t)} f(t) - \alpha_n \, \mathrm{d}w(t) \otimes \mathrm{d}w(t)) \\ \frac{\partial}{\partial t} w(t) = \Delta_{g(t)} w(t) - \langle \nabla w(t), \nabla f(t) \rangle_{g(t)}, \\ \frac{\partial}{\partial t} f(t) := h = \frac{v^{\alpha}{}_{\alpha}}{2} + \frac{n}{2\tau} = -\Delta_{g(t)} f - R_{g(t)} + \alpha_n |\nabla w(t)|^2_{g(t)} + \frac{n}{2\tau}, \\ \frac{\mathrm{d}}{\mathrm{d}t} \tau = -1, \end{cases} \tag{3.2}$$

has a solution in $M \times [0,T)$. To find a solution to (3.2) we consider a solution of the *backward heat equation* $\frac{\partial}{\partial t} f(t) = -\Delta_{g(t)} f(t) + |\nabla f(t)|^2 - R_{g(t)} + \alpha_n |\nabla w(t)|^2_{g(t)} + \frac{n}{2\tau}$ along the extended Ricci flow in $M \times [a,b]$, which is obtained as follows. Let $[a,b]$ be a sub-interval of $[0,T)$ and $(g(t), w(t))$ satisfying the extended Ricci flow equation $\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}_{g(t)} + 2\alpha_n \, \mathrm{d}w(t) \otimes \mathrm{d}w(t)$ with $\frac{\partial}{\partial t} w(t) = \Delta_{g(t)} w(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t} \tau = -1$ in $[a,b]$. Take $z(t) := (4\pi\tau(t))^{-\frac{n}{2}} e^{-f(t)}$ and define $s = T - t$. Since $\Delta_g z = (|\nabla f|^2 - \Delta_g f) z$, one has

$$\frac{\partial}{\partial s} z = -z \frac{n}{2\tau} \frac{\mathrm{d}}{\mathrm{d}s} \tau + z \frac{\partial}{\partial t} f = z(-\Delta_g f + |\nabla_g f|^2 - R_g + \alpha_n |\nabla w|^2_g) = \Delta_g z - R_g z + \alpha_n |\nabla w|^2_g z$$

which is a parabolic equation in $M \times [a,b]$. It guarantees the existence of $f(t)$ along the extended Ricci flow in $M \times [a,b]$. Now, let $\{\phi_t\}_{t \in [a,b]}$ be the one-parameter family of diffeomorphisms generated by $\{-\nabla_{g(t)} f(t)\}_{t \in [a,b]}$, with $\phi_a = \mathrm{Id}$. By setting $\widetilde{g}(t) := \phi_t^* g(t)$, $\widetilde{w}(t) := \phi_t^* w(t)$, $\widetilde{f}(t) := \phi_t^* f(t)$, we have

$$\frac{\partial}{\partial t} \widetilde{g}(t) = \phi_t^* (\frac{\partial}{\partial t} g(t)) + \phi_t^* \mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t} \phi_t} g(t) = -2(\mathrm{Ric}_{\widetilde{g}(t)} + \mathrm{Hess}_{\widetilde{g}(t)} \widetilde{f}(t) - \alpha_n \, \mathrm{d}\widetilde{w}(t) \otimes \mathrm{d}\widetilde{w})$$

as well

$$\frac{\partial}{\partial t} \widetilde{w} = \phi_t^* \Big(\frac{\partial}{\partial t} w \Big) + \phi_t^* \mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t} \phi_t} w = \phi_t^* \Big(\Delta w \Big) - \phi_t^* \mathcal{L}_{\left(\nabla_{g(t)} f(t) \right)} w = \Delta_{\widetilde{g}} \widetilde{w} - \langle \nabla \widetilde{w}, \nabla \widetilde{f} \rangle_{\widetilde{g}}.$$

Moreover,

$$\frac{\partial}{\partial t} \widetilde{f}(t) = \phi_t^* (\frac{\partial}{\partial t} f(t)) + \phi_t^* \mathcal{L}_{\frac{\mathrm{d}}{\mathrm{d}t} \phi_t} f(t) = -\Delta_{\widetilde{g}(t)} \widetilde{f}(t) - R_{\widetilde{g}(t)} + \alpha_n |\nabla_{\widetilde{g}} \widetilde{w}|^2 + \frac{n}{2\tau}.$$

The first and third equations imply $\frac{\widetilde{g}^{\alpha\beta} \frac{\partial}{\partial t} \widetilde{g}_{\alpha\beta}}{2} - \frac{\partial}{\partial t} \widetilde{f} + \frac{n}{2\tau} = 0$. Hence, $(\widetilde{g}(t), \widetilde{f}(t), \widetilde{w}(t))$ is a solution to (3.2).

Now, observe that

$$\begin{aligned} \xi g^{\alpha\beta} - \tau v^{\alpha\beta} &= -g^{\alpha\beta} - \tau \big(-2(R^{\alpha\beta} + \nabla^{\alpha} \nabla^{\beta} f - \alpha_n \nabla^{\alpha} w \nabla^{\beta} w) \big) \\ &= 2\tau (R^{\alpha\beta} + \nabla^{\alpha} \nabla^{\beta} f - \alpha_n \nabla^{\alpha} w \nabla^{\beta} w - \frac{1}{2\tau} g^{\alpha\beta}), \end{aligned}$$

and then by Corollary 3.3 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_\infty^{\alpha_n}\Big(\frac{\partial}{\partial t}\widetilde{g},\frac{\partial}{\partial t}\widetilde{f},\frac{\partial}{\partial t}\widetilde{w},\frac{\mathrm{d}}{\mathrm{d}t}\tau\Big)=2\tau\int_M\Big[\big|\operatorname{Ric}_{\widetilde{g}}+\operatorname{Hess}_{\widetilde{g}}\widetilde{f}-\alpha_n\,\mathrm{d}w(t)\otimes\mathrm{d}w(t)-\frac{1}{2\tau}\widetilde{g}\big|^2 \\ +\alpha_n(\Delta_{\widetilde{g}}\widetilde{w}-\langle\widetilde{\nabla}\widetilde{w},\widetilde{\nabla}\widetilde{f}\rangle_{\widetilde{g}})^2\Big]\widetilde{u}\,\mathrm{dV}_{\widetilde{g}},$$

where $\widetilde{u}=(4\pi\tau)^{-\frac{n}{2}}e^{-\widetilde{f}}$. So, $\mathcal{W}_\infty^{\alpha_n}$ is constant in time if and only if $(\widetilde{g}(t),\widetilde{w}(t))$ is a gradient shrinking soliton to the extended Ricci flow on M with potential function $\widetilde{f}(t)$.

The variation of $\mathcal{W}_\infty^{\alpha_n}$ under preserving-measure $v\,\mathrm{dV}$ (see Corollary 3.2) from which one has

$$\delta\mathcal{W}_\infty^{\alpha_n}(v,h,\vartheta,\xi) \\ =\int_M\Big[(\xi g^{\alpha\beta}-\tau v^{\alpha\beta})(R_{\alpha\beta}+\nabla_\alpha\nabla_\beta f-\alpha_n\nabla_\alpha w\nabla_\beta w-\frac{1}{2\tau}g_{\alpha\beta})+2\tau\alpha_n\vartheta(\Delta_g w-\langle\nabla f,\nabla w\rangle)\Big]v\,\mathrm{dV} \\ +\int_{\partial M}\Big[\xi(2H+e_0f)+2\alpha_n\tau\vartheta e_0w-\tau\big(A^{ij}v_{ij}+v^{00}(H+e_0f)\big)\Big]v\,\mathrm{dA}\,.$$

Now changing (2.30) to

$$\begin{cases}\frac{\partial}{\partial t}u=-\Delta_g u+Ru-\alpha_n|\nabla w|^2u+\frac{n}{2\tau}u,\\ \frac{\mathrm{d}}{\mathrm{d}t}\tau=-1.\end{cases}\tag{3.3}$$

satisfying the boundary condition $e_0u=Hu$ with $u=e^{-f}$, then we proceed as in the proof of Proposition 2.19 in order to obtain by Corollary 3.2

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}_\infty^{\alpha_n}=2\int_M\tau\Big[\big|\operatorname{Ric}+\operatorname{Hess}f-\alpha_n\,\mathrm{d}w\otimes\mathrm{d}w-\frac{1}{2\tau}g\big|^2+\alpha_n\big(\Delta_g w-\langle\nabla w,\nabla f\rangle\big)^2\Big]v\,\mathrm{dV} \\ +2\int_{\partial M}\tau\Big(\frac{\partial}{\partial t}H-2\langle\widehat{\nabla}f,\widehat{\nabla}H\rangle+\mathcal{A}(\widehat{\nabla}f,\widehat{\nabla}f)+2R^{0i}\widehat{\nabla}_if-\frac{1}{2}\nabla_0R-HR_{00}-\frac{H}{2\tau}\Big)v\,\mathrm{dA}\,.$$

In particular, by considering M compact without boundary, we recover the result by List [Lis08, Thm. 6.1].

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